FOUNDATIONS PAPER

Imagine Math®

Executive Summary

The Challenge

Research has consistently documented the influential role mathematics has in students' academic success and future career opportunities (National Research Council, 2012). There is widespread agreement that students' early mathematics knowledge predicts their success in algebra and overall mathematics achievement in high school (Baroody & Purpura, 2017; Powell & Fuchs, 2017; Siegler et al., 2012). The United States continues to lag behind other top-performing education systems, such as Singapore and the Netherlands in mathematics (Trends in International Mathematics and Science Study, 2019). Data from the most recent National Assessment of Educational Progress (NAEP, 2019) indicate that only 41% of fourth-grade students and 34% of eighth-grade students demonstrated proficiency in mathematics. Such disparities in student performance illustrate why the traditional one-size-fits-all approach to teaching mathematics is no longer adept at meeting students' needs. Diversity inside classrooms necessitates the design of mathematics curricula that provide developmentally appropriate instruction for students who come from diverse social, cultural, linguistic, and ability backgrounds, to ensure that all students have fair opportunities to succeed.

The Solution

A growing number of schools and districts are turning to online and blended-learning programs to enhance mathematics curricula and better serve their students. **Imagine Math** offers a comprehensive supplemental solution designed for students in prekindergarten (PreK) through high school. This supplemental program provides adaptive, developmentally appropriate instruction that focuses on building students' conceptual understanding of mathematics. Imagine Math believes all students deserve access to rigorous mathematics instruction and ensures their experiences in the program empower them to explore mathematics in deep and meaningful ways.

In Imagine Math, students begin by taking a computer-based adaptive Benchmark Test to screen and determine their readiness for mathematics instruction. This enables **Imagine Math** to place each student on a customized learning pathway. Two additional Benchmark Tests are embedded throughout the year, which allows teachers to continue to monitor progress and growth in student achievement. The adaptive algorithms and ongoing assessments powered by the Quantile® Framework for Mathematics differentiate instruction so that learning is intentionally scaffolded up to grade-level content and beyond. The instructional content is closely aligned with national and state standards, as well as school and district learning goals. When used with fidelity, the program accelerates student achievement on a wide array of measures, including the Northwest Evaluation Association (NWEA) Measures of Academic Progress (MAP) mathematics assessments, State of Texas Assessments of Academic Readiness (STAAR) mathematics test, Partnership for Assessment of Readiness for College and Careers (PARCC) assessment, and Smarter Balanced Assessment Consortium (SBAC) math assessment.

Imagine Math transforms learning by:

1. **Designing developmentally appropriate learning environments that promote mastery of grade-level content.**

   Imagine Math's developmentally appropriate learning environments are intentionally designed to support students' conceptual understanding of mathematics. These environments target students in **PreK–Grade 2** and **Grade 3–High School (including Algebra I and Geometry).** Within each environment, lessons contain a variety of models and representations, incorporate rich academic language, and promote conceptual understanding. These lessons also nurture students' problem-solving skills, reasoning, and real-world application abilities.

2. **Providing adaptive learning pathways, which include ongoing scaffolded support and immediate feedback to differentiate learning.**

   The use of scaffolding and adaptive feedback differentiates learning so that instruction falls within each student's zone of proximal development. Built-in scaffolds (e.g., digital mathematics manipulatives; an interactive, bilingual glossary; instructional games; support from certified Live Teachers) and immediate feedback emphasize mastery, not accuracy or performance. If needed, prerequisite lessons are inserted to help students acquire the necessary background information to successfully engage with the content. Instruction is always scaffolded appropriately, and never watered down, to maintain the academic rigor of each lesson. Lessons are supported in English and Spanish to promote language development and effective communication skills.

3. **Integrating unique motivational elements that foster curiosity, interest, and engagement.**

   Imagine Math recognizes that students are motivated in different manners. Therefore, the program comprises a unique motivation system based on a single idea—rewarding effort and accomplishment. Students explore highly engaging content that sparks curiosity and interest, as well as perseverance.

Together, these features undergird the development of Imagine Math. This program strives to promote curious, confident, and competent mathematicians by thoughtfully translating research into practice to create learning environments that are proven to advance learning. Students' engagement with Imagine Math strengthens their understanding of grade-level content, helping them develop confidence in their abilities and a love for mathematics.

Imagine Math's Theory of Action

Imagine Math has a well-specified theory of action that explains how the intervention is likely to improve learning outcomes. Figure 1 displays the relationship between Imagine Math's inputs, activities, and desired outcomes. The model outlines the resources (e.g., devices, teacher buy-in) needed to effectively implement this solution (e.g., 2–3 lessons per week, offline resources) to produce outputs that lead to short-term (e.g., increased engagement, growth on Benchmark Tests) and long-term outcomes (e.g., increased mathematics proficiency on state standardized tests, self-confidence).

[1] *Quantile and Quantile Framework are registered trademarks of MetaMetrics, Inc., and are registered in the United States and abroad.*

Figure 1. *Imagine Math's Theory of Action*

Program Inputs	Program Inputs: District
Research-based, standards-aligned supplemental program to provide meaningful practice and promote mastery of grade-level content Scaffolded support and informative feedback to make learning accessible for all students Embedded motivation system to engage learners and encourage perseverance Diagnostic Benchmark Tests for placement and on-going formative assessments for progress monitoring Actionable reports that drive instruction for a whole class or individual students Flexible model for delivery Professional development, training, and support	Access to **Imagine Math** instructional content via site license Technology: networked computers or mobile devices, headsets, and supporting hardware and software School and district infrastructure to support technology use Teacher buy-in and readiness to adopt technology School implementation plan School or district learning goals

Classroom Activities	Outputs
Student Activities: Spend at least 45 minutes (or 2–3 lessons) per week (PreK–Grade 2)† Spend 60–90 minutes (or 2–3 lessons) per week (Grade 3–High School)†† Pass 30 lessons before the end of the school year Engage in offline resources • Printable worksheets† • Printable worksheets; Application Tasks; Journaling Pages†† **Teacher Activities:** Implement blended learning model(s): whole-class instruction, computer lab, in-class rotation, intervention, extended learning (at home, after school, summer school, etc.) Use actionable data to monitor student progress and plan for differentiated instruction	Tracked in Imagine Math data reports: **Implementation Metrics** • Number of districts, schools, students, and teachers **Progress Metrics** • Number of lessons completed • Number of problems completed • Percent of tokens earned† • Number of Math Helps†† used • Number of Live Help Sessions†† used **Student Usage** • Number of total students using or enrolled • Number of active students using Imagine Math at school and/or at home • Average student usage, percentage of goal **Student Progress Lessons** • Average weekly mathematics time • Number of lessons completed • Number of lessons passed **Student Progress Assessments** • Number of assessments completed • Quantile measure • Student performance level, percentile rank, and instructional grade level

Short-term Outcomes	Long-term Outcomes
Student Outcomes: Students exhibit increased engagement as measured by usage of and progress through Imagine Math Students increase mathematics proficiency as evidenced by their performance on Imagine Math assessments	**Student Outcomes:** Students increase mathematics proficiency on nationally normed or standardized assessments Students increase academic achievement in other subject areas Students develop motivation, self-efficacy, and self-confidence to learn mathematics **Teacher Outcomes:** Teachers feel prepared to implement Imagine Math in their classrooms Teachers build understanding of students' mathematical thinking

† Specific to the PreK–Grade 2 learning environment.

†† Specific to the Grade 3–High School learning environment.

Imagine Math's Research-Based Principles

To create these short- and long-term outcomes, **Imagine Math** prioritizes the best empirical, pedagogical, and theoretical research to create developmentally appropriate instruction that improves students' mathematics achievement. **Imagine Math** thoughtfully considers the most effective approaches for teaching mathematics to students at different points in their development. Accordingly, each learning environment is grounded in research-based principles that are considered most appropriate for their age.

In the **PreK–Grade 2** learning environment, **Imagine Math** integrates the following research-based principles.

- Incorporate play-based pedagogies that scaffold learning and ensure all students have access to grade-level mathematics content.
- Support students' conceptual understanding of mathematics through spiral learning and meaningful practice.
- Provide a positive digital learning environment that integrates real-world situations to help children grow as mathematicians.
- Foster a lifelong love of mathematics among early learners by increasing motivation and engagement, and promoting curiosity and confidence.
- Differentiate learning by providing informative feedback and adaptive, diagnostic assessments.

In the **Grade 3–High School** learning environment, **Imagine Math** integrates the following research-based principles.

- Provide scaffolded instruction that promotes mastery of grade-level content in number and operations; algebra; geometry; measurement; and data, probability, and statistics.
- Integrate research-based mathematics teaching practices that encourage problem solving, reasoning, and real-world application.
- Promote mathematical discourse to help students develop effective communication skills and a deep understanding of mathematics.
- Ensure all learners receive equitable and accessible mathematics instruction.
- Utilize intrinsic and extrinsic motivational strategies to foster active engagement, collaboration, and perseverance.
- Differentiate instruction by offering informative feedback and adaptive assessments, while providing actionable data to inform mathematics teaching and improve student performance.

The following foundations research paper illustrates how **Imagine Math**, a supplemental mathematics learning solution, translates research into practice. This paper summarizes current literature in mathematics education, provides research-based recommendations for effective mathematics instruction, and explains how Imagine Math integrates these instructional recommendations to advance learning. First, the **Imagine Math PreK–Grade 2 (IM PreK–2)** learning environment is discussed, followed by the **Imagine Math Grade 3–High School (IM 3+)** learning environment.

Imagine Math PreK–Grade 2 Learning Environment

The **Imagine Math PreK–Grade 2 (IM PreK–2)** learning environment is designed to support early learners' innate curiosity and interest in mathematics. Students begin with an introduction to the characters in **Imagine Math**. Prior to completing one full lesson, students familiarize themselves with the environment by learning how to click and drag onscreen objects, navigate lessons, and explore the different areas on the map. Then, students take Benchmark Test 1. This enables **IM PreK–2** to place each student in a customized learning pathway. Two additional Benchmark Tests are embedded throughout the year, which allows teachers to monitor student progress and growth. The adaptive algorithms and ongoing assessments powered by the Quantile Framework for Mathematics differentiate instruction so that learning is intentionally scaffolded up to grade-level content. The instructional content within each lesson is closely aligned with national and state standards so that the **IM PreK–2** curriculum complements what students learn during core mathematics instruction.

Visual models, memorable songs, and contextualized vocabulary are at the heart of all **IM PreK–2** lessons. Lessons contain a series of activities, with each activity consisting of several different exercises. Within each lesson, there are activities that focus on a specific mathematics concept or skill to promote mastery of grade-level concepts. There are also activities that revisit previously learned material to help students make stronger connections across mathematics concepts. As students work through a lesson and explore the virtual world of **Imagine Math**, they learn from lovable characters (Figure 2), animations, songs, and captivating storylines. Students have the autonomy to complete activities and earn tokens, which maintains engagement and increases their motivation to learn. For instance, after completing a lesson, students can visit Treasure Island, the Music Hall, the Info Center, or the Fair (Figure 3).

- **Treasure Island**—Students visit Treasure Island to earn tokens they have missed in previous lessons.
- **Music Hall**—Students can watch and sing along to songs that are part of the lessons.
- **Info Center**—Students can learn more about the characters who appear in **IM PreK–2**.
- **The Fair**—Students can access intervention games (Figures 4 and 5), which focus on building prerequisite skills and mathematics fluency in fun and effective ways.

In addition, students have access to a vast library of printable worksheets to optimize student learning, online and offline (Figures 6 and 7). These are available in both English and Spanish.

Figure 2. *IM PreK–2 characters*

Figure 3. *Locations on the map*

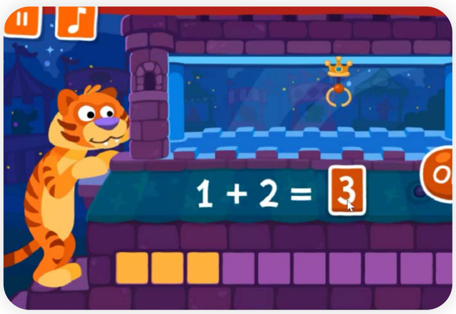


Figure 4. *Intervention game*

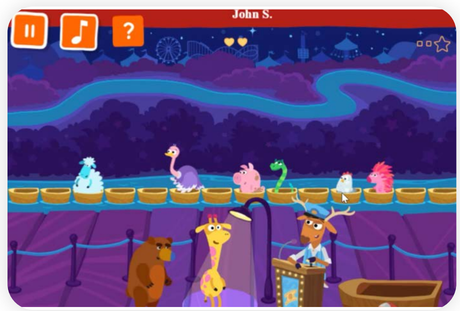


Figure 5. *Intervention game*

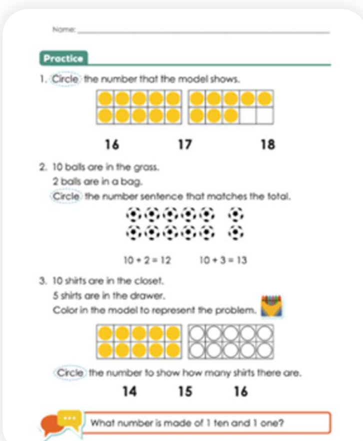


Figure 6. *Printable worksheet (English)*

Figure 7. *Printable worksheet (Spanish)*

In the **PreK–Grade 2** learning environment, **Imagine Math** integrates the following research-based principles.

- Incorporate play-based pedagogies that scaffold learning and ensure all students have access to grade-level mathematics content.
- Support students' conceptual understanding of mathematics through spiral learning and meaningful practice.
- Provide a positive digital learning environment that integrates real-world situations to help children grow as mathematicians.
- Foster a lifelong love of mathematics among early learners by increasing motivation and engagement, and promoting curiosity and confidence.
- Differentiate learning by providing informative feedback and adaptive, diagnostic assessments.

Principle 1: Incorporate play-based pedagogies that scaffold learning and ensure all students have access to grade-level mathematics content.

WHAT THE RESEARCH SAYS:

Play-based learning has been widely adopted as the cornerstone of early childhood education (Edwards, 2017; Yin et al., 2021). Specifically, research has found that guided play-based learning, which unifies play with instruction, supports young children's cognitive, emotional, and social development (Pyle & Alaca, 2018; Pyle & Danniels, 2017). In this approach, teachers develop learning objectives and provide support to reinforce children's understanding of concepts and skills (Bubikova-Moan et al., 2019; Miller, 2018; Pyle & Deluca, 2017; Weisberg et al., 2013). Research shows that guided play-based learning is more effective than free play and direct instruction (Han et al., 2010; Honomichl & Chen, 2012; Pyle & Danniels, 2017) and can be used to support students' development of numeracy skills (Miller, 2018; Vogt et al., 2018) and problem-solving skills (Taylor & Boyer, 2020).

Play-based learning shares many commonalities with the **Universal Design for Learning (UDL)** framework (Center for Applied Special Technology [CAST], 2018) in that both recognize the need for differentiated instruction. UDL is an instructional framework designed to provide all students with equal opportunities to succeed. This framework recommends that instruction be organized around three fundamental principles (providing *multiple means of representations; action and expression;* and *engagement*) (CAST, 2018) to adequately meet the cognitive, physical, and socio-emotional needs of young learners (Conn-Powers et al., 2006). In the early childhood education sector, scaffolding children through play (e.g., modeling) is an important component of play-based pedagogies and learning (Keung & Cheung, 2019; Pyle & Danniels, 2017). The term **adaptive scaffolding** is used to describe support that extends, enriches, and intensifies learning by providing helpful hints, cues, or adapted activities (National Association for the Education of Young Children [NAEYC], 2020). Research affirms that adaptive scaffolding can make learning **accessible** to all students, enabling them to successfully engage with grade-level content (National Council of Teachers of Mathematics [NCTM], 2014a), and improve their academic outcomes (Belland et al., 2017; Gersten et al., 2009; Hudson et al., 2006; Smith et al., 2016).

RESEARCH-BASED RECOMMENDATIONS:

- Incorporate guided **play-based learning** pedagogies into mathematics teaching to promote engagement and active learning. Identify specific learning objectives and design instruction around those goals to help students learn the target concept or skill (Pyle & Danniels, 2017; Vogt et al., 2018).
- Ground mathematics teaching in **Universal Design for Learning** principles to support all students regardless of race, gender, socioeconomic status, language proficiency, learning-disability status, or other social or cultural factors (CAST, 2018; Kieran & Anderson, 2019).
    - Create learning opportunities that are relevant and meaningful.
    - Integrate multiple forms of *representation* to reduce barriers to print and ensure information is equally perceptible to all students.
    - Incorporate new vocabulary and frequent opportunities to hear and use vocabulary to ensure comprehensibility for all learners.

- Encourage the use of diverse tools and multimedia technologies to *express* and communicate understanding of critical ideas.
- Provide corrective feedback that is clearly and explicitly connected to high standards. Feedback should capitalize on mistakes as opportunities to learn.
- Encourage persistence, *engagement*, and motivation.

- Make learning **accessible** to all students by providing **adaptive scaffolding** that promotes mastery of grade-level content (Gottlieb, 2016; Lei et al., 2020). Learning opportunities should also fall within the student's zone of proximal development (Vygotsky, 1978).

HOW IMAGINE MATH PREK–GRADE 2 INTEGRATES THESE RECOMMENDATIONS:

IM PreK–2 believes students learn best through **play-based** activities. This learning environment provides fun, play-based lessons that incorporate engaging gamified elements. **IM PreK–2** lessons have two settings: one set in the daytime, where students experience real-world situations and discover mathematics around them (Figures 8 and 9); and a second set in the nighttime, where students explore a fantasy dream world where fun, imaginative things can happen (Figures 10 and 11). Each lesson presents a new mathematics concept or extends students' understanding of a previously learned concept through appealing animations, dynamic characters, songs, storylines, and interactive games.

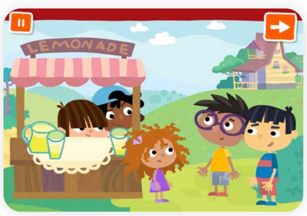


Figure 8. *Daytime*

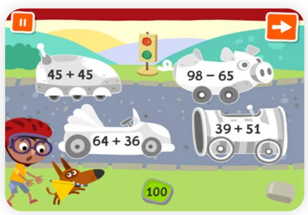


Figure 9. *Daytime*

Figure 10. *Nighttime*

Figure 11. *Nighttime*

IM PreK–2 thoughtfully integrates the **Universal Design for Learning** principles so that learning falls within each student's zone of proximal development. **IM PreK–2** aims to reach the widest student audience by presenting activities in different ways. Lessons provide:

- **Multiple means of representation** present information in different modalities, such as visual text (e.g., word problems presented in English or Spanish), audio (presented in English or Spanish), virtual manipulatives (e.g., counters, pattern blocks), visual models (e.g., segment model, bar model), and visual representations (e.g., number line, ruler, bar graph). For instance, when students explore number magnitude concepts, lessons incorporate multiple forms of representation so all students have opportunities to make sense of the information in meaningful ways. These representations include the use of a number line to compare sums (Figure 12); base-10 manipulatives to determine which symbol (>, <, =) makes the expression true (Figure 13); a bar model (Figure 14); and a segment model when solving a compare word problem type (Figure 15).

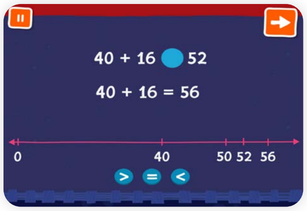


Figure 12. *G.1-Lesson 106*

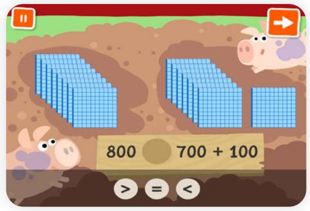


Figure 13. *G.2-Lesson 055*

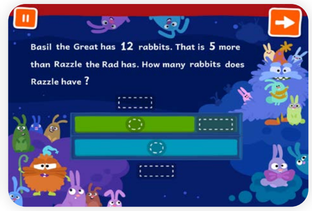


Figure 14. *G.2-Lesson 079*

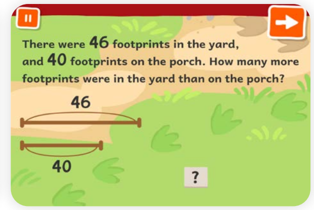


Figure 15. *G.2-Lesson 038*

- **Multiple means of action and expression** encourage students to communicate their understanding in a variety of ways. Students learn how to express their mathematical thinking by listening to the dialogue between the characters, and by utilizing drag-and-drop responses, virtual manipulatives, and printable resources in English or Spanish (Figure 16). These resources not only promote language development by targeting specific mathematics vocabulary, but they also include opportunities for students to communicate through text or orally with their peers. For instance, this printable resource asks students to "Describe how a cylinder and a cone are similar and different."

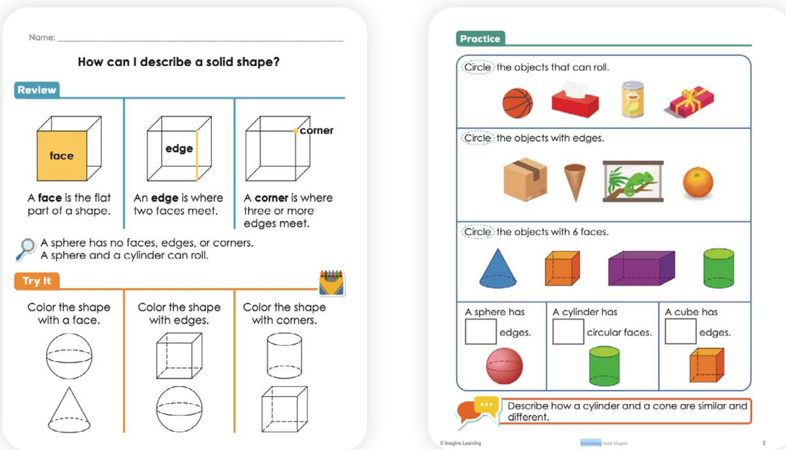


Figure 16. *G.1-Describe Solid Shapes*

- **Multiple means of engagement** embed different strategies that captivate students' attention and maintain their motivation. To accomplish this, students find themselves in a virtual world much like their real world, which allows them to make connections between the mathematics they are learning and their everyday lives. Animated characters and the stories they act out motivate students to learn. The adaptive learning pathways, scaffolding, constant feedback, and tokens that students can earn encourage them to persevere through challenging tasks.

IM PreK–2 believes all students are capable of success with grade-level content. The program provides adaptive **scaffolding** to make learning **accessible** for all students. To maintain the academic rigor of each lesson, a unique array of scaffolds is integrated throughout the lessons. This helps balance the level of challenge with support provided to students.

- **Language Support.** Students' home language is viewed as a valuable attribute. Lessons are fully available in English or Spanish. The dialogue between characters also provides language support by incorporating academic language and mathematics vocabulary words into conversations. Providing support in students' home language helps reduce the cognitive load and allows them to focus on the mathematical concepts and skills being addressed in each activity.

- **Multimedia Support.** Students can utilize virtual manipulatives (e.g., counters), animations, multimedia response options (e.g., drag and drop), audio support (e.g., feedback provided verbally), and visual support (e.g., text highlighting) (Figure 17).
- **Printable Resources.** Printable resources are available for offline use. These worksheets can be used to reteach, intervene, or focus on foundational skill mastery. They are available in English and Spanish (Figure 18).
- **Immediate Feedback.** Immediate feedback is designed to scaffold learning, reinforce correct responses, and address misconceptions. For instance, bar models and step-by-step directions assist students as they solve word problems (e.g., "*Ruby was outside for 2 hours* [program moves '2 hours' to its correct location]. *She spent some of the time reading, some of the time swinging, and some of the time climbing the tree* [program moves labels to correct location on model]. *Put the time Ruby spent doing each activity in the model.*" (Figure 19). Additionally, feedback is intentionally designed to be encouraging. Rather than identifying the correct answer and moving on, **IM PreK–2** promotes a positive mathematical mindset and encourages learners to persevere by verbalizing phrases such as "*Let's give it another try!*"

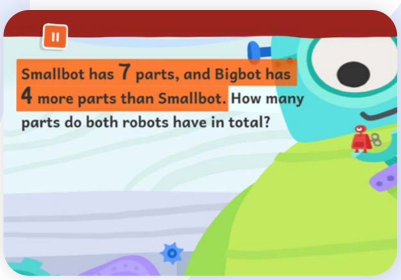
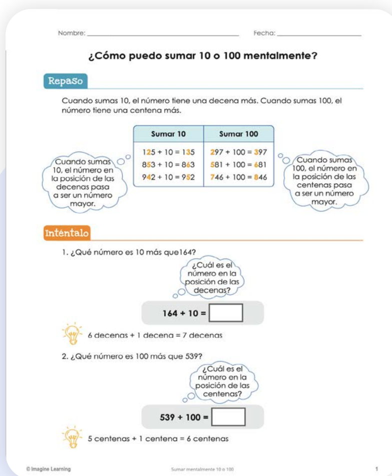


Figure 17. *G.1-Lesson 075*



Nombre: ______ Fecha: ______

¿Cómo puedo sumar 10 o 100 mentalmente?

Repaso

Cuando sumas 10, el número tiene una decena más. Cuando sumas 100, el número tiene una centena más.

Sumar 10	Sumar 100
125 + 10 = 135	297 + 100 = 397
853 + 10 = 863	581 + 100 = 681
942 + 10 = 952	746 + 100 = 846

Cuando sumas 10, el número en la posición de las decenas pasa a ser un número mayor.

Cuando sumas 100, el número en la posición de las centenas pasa a ser un número mayor.

Inténtalo

1. ¿Qué número es 10 más que164?

¿Cuál es el número en la posición de las decenas?

164 + 10 = ☐

6 decenas + 1 decena = 7 decenas

2. ¿Qué número es 100 más que 539?

¿Cuál es el número en la posición de las centenas?

539 + 100 = ☐

5 centenas + 1 centena = 6 centenas

© Imagine Learning Sumar mentalmente 10 o 100 1

Figure 18. *G.2-Add 10 or 100*

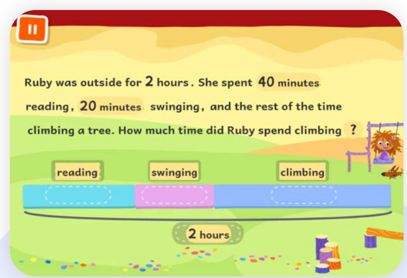


Figure 19. *G.2-Lesson 088*

Principle 2: Support students' conceptual understanding of mathematics through spiral learning and meaningful practice.

WHAT THE RESEARCH SAYS:

To build their confidence and understanding of mathematics concepts, children need ongoing opportunities to make connections between new information and prior knowledge. This idea dates back to Bruner's (1960) concept of a **spiral curriculum**, which refers to a curriculum in which students have repeated opportunities to revisit a topic in greater depth each time. Topics are addressed in increasing levels of difficulty to present new learning opportunities and promote students' application of previous knowledge and skills (Harden & Stamper, 1999). The concept of a spiral curriculum shares similarities with distributed practice.

Distributed practice promotes mastery and long-lasting retention of concepts by prioritizing the quality of content addressed over the course of a student's educational experience, rather than the quantity (Wiseheart et al., 2019). Research has documented the effectiveness of distributed practice on students' mathematics learning (Foster et al., 2015), and studies show that distributed practice leads to significantly higher fact fluency growth rates than a mass practice approach (Schutte et al., 2015). The cyclic nature of a spiral curriculum combined with distributed practice allows for integration and continuity within and across mathematics topics, an important component of deepening students' conceptual understanding.

As children actively make sense of the world around them, mathematics becomes meaningful. Students benefit from learning opportunities that are meaningful and relevant, enable them to practice a range of concepts, make connections across those concepts, and engage them in conceptual-focused activities. In fact, studies show that **meaningful practice** can predict children's mathematics proficiency (Sigmundsson et al., 2013). The concrete-representational-abstract framework (Bruner & Kenney, 1965) is a well-documented approach for supporting students' conceptual understanding. In this approach, support is gradually faded as students begin to master key mathematics concepts (Agrawal & Morin, 2016; Bouck et al., 2018). Students begin by solving problems using concrete objects, followed by visual representations, and finally solving problems abstractly. Experts recommend children have opportunities to engage in meaningful practice with number sense, whole-number operations, early algebraic thinking, geometry, and measurement and data.

Number Sense

Children's early number sense is one of the strongest predictors of later mathematics achievement (Nguyen et al., 2016). **Number sense** refers to the ability to understand, represent, and reason flexibly about the relationships between numbers (Green & Towson, 2020). Research shows that the development of number sense occurs in a series of sequential phases. Experts recommend that instruction teach concepts in the following order: 1) subitizing, or the ability to recognize small collections of objects quickly without counting; 2) verbal and object counting, including one-to-one correspondence and cardinality; 3) knowing one more/one less in the sequence; 4) counting objects by groups and using numerals to describe the quantities; and 5) number magnitude, or the ability to mentally and symbolically compare, order, and estimate numbers (Clements & Sarama, 2021; Powell & Fuchs, 2012; Richardson, 2012; Witzel et al., 2013). It has been noted that the "primary cause of problems with the basic combinations, especially among children at risk for or already experiencing learning difficulties, is the lack of opportunity to develop number sense during the preschool and early school years" (Baroody et al., 2009, p. 69). To mitigate these challenges, early childhood mathematics curricula should emphasize counting, number magnitude, and place value to strengthen students' understanding of number relationships (Powell & Fuchs, 2012).

- **Counting** is associated with number sequence, one-to-one correspondence, and cardinality (Clements & Sarama, 2014). Through their early experiences, children develop an understanding of **number sequence,** or the names and ordered list of number words. Then, children begin to connect the sequence through **one-to-one correspondence** between objects in a collection and the collection being counted. They also learn the **cardinality principle,** or that the last counting word indicates how many objects are in the collection. Collectively, these concepts are important for developing students' number sense and building a foundational understanding of part-whole relations and additive composition (Sedaghatjou & Campbell, 2017).

- **Number magnitude** involves the ability to "comprehend, estimate, and compare the sizes of numbers" (Fazio et al., 2014, p. 54). Students make meaning of concepts like "more than" and "less than" through experiences discriminating the magnitude of non-symbolic representations (e.g., objects) and symbolic representations (e.g., numbers) (Siegler, 2016). As children learn to compare and order numbers, estimation becomes an important part of understanding magnitude. It helps students develop a mental number line and improves their overall mathematics achievement (Scalise & Ramani, 2021; Siegler, 2016).
- **Place value** knowledge plays an important role in understanding the underlying structure of the base-10 system, as well as whole-number and rational-number computation (Richardson, 2012; Witzel et al., 2013). Yet, many students lack a conceptual understanding of place value (Hartnett, 2018; MacDonald et al., 2018). To strengthen students' understanding, it is critical that instruction emphasize a range of mathematics concepts that underpin place value, "including the structure of our number system; reading, writing and ordering numbers; the relationship between the places; the role of the decimal point and use of zero" (Hartnett, 2018, p. 36). In addition, attending to unit coordination and teaching students to unitize (or group) is important for understanding multi-digit place value (Brendefur et al., 2018; MacDonald et al., 2018).

Whole-Number Operations

A strong number sense and understanding of the base-10 system influence students' **whole-number computation skills** (Hickendorff et al., 2019). In the early elementary grades, effective instruction focuses on students' development of additive reasoning skills, or their understanding of part-whole relations (Vergnaud, 1982). The emphasis is on one unit, in which students learn how groups are combined successively at one level. As students develop additive reasoning skills, they learn how numbers can be composed and decomposed (Richardson, 2012), develop conceptual strategies to solve single-digit addition and subtraction problems (Clements & Sarama, 2021), and develop computational fluency (Bay-Williams & Kling, 2014). Researchers urge against rote drill practice and memorized facts because students need to understand the *why* behind *what* they are computing (Clements & Sarama, 2021). This is important because research shows mastering single-digit computation and developing fluency with facts are related to students' proficiency with multi-digit addition and subtraction (Hickendorff et al., 2019). Providing opportunities for students to develop strong additive-reasoning skills is critical for their transition to multiplicative reasoning and future work with multi-digit operations, rational numbers, proportional reasoning, and algebra (Ebby et al., 2021).

Algebra

Algebra is often described as a gatekeeper to opportunities—namely, advanced mathematics, college access, and prospective careers (Moses & Cobb, 2001). Studies provide compelling evidence that young children can successfully engage in algebraic thinking (Blanton et al., 2015; Cai & Knuth, 2011; Chimoni et al., 2018; Knuth et al., 2016). In fact, their understanding of early algebra concepts, such as equivalence, predicts their algebraic thinking skills in later grades (Hornburg et al., 2021; Matthews & Fuchs, 2020). To promote students' early algebraic thinking, researchers recommend that instruction emphasize a range of early algebra concepts, such as arithmetic, equivalence, and patterns (Blanton et al., 2019; Carraher & Schliemann, 2018; Kaput, 2008; Stephens et al., 2015).

- **Arithmetic** includes the ability to reason about the operation, use and apply the fundamental properties of number and operation (e.g., commutative property of addition), and use symbols to represent unknown or varying quantities (Banerjee & Subramaniam, 2012; Blanton et al., 2017; Kieran, 2018; Pang & Kim, 2018).
- Understanding **equivalence** and the relational meaning of the equal sign is critical to students' success in algebra (Knuth et al., 2016). However, students harbor serious misconceptions about the equal sign (Bryd et al., 2015; Stephens et al., 2013). Many regard the equal sign as a symbol of action, rather than a symbol denoting the relationship between two quantities (Powell et al., 2020). Studies show students who interpret the equal sign as a symbol of action perform lower on algebra tasks, with more profound negative consequence across grade levels (Byrd et al., 2015).
- **Patterns** are a powerful means of stimulating algebraic thinking (Miller et al., 2016; Rittle-Johnson et al., 2015). Repeating patterns (a single repeating item in a sequence) challenge students to attend to regularity, repetition, and relationships. Researchers contend that pattern activities are a critical component of early algebra (Greenes et al., 2001; Papic et al., 2011) and data show an association between early pattern activities and mathematical abilities (Kidd et al., 2014; Pasnak, 2017).

Geometry

Experts agree that drawing on a play-based approach to teach geometry can help students construct critical ideas about shapes and other geometry concepts (Clements & Sarama, 2021). **Geometry** is commonly described as "a network of concepts, ways of reasoning, and representation systems" that challenges students to explore and analyze shapes and space (Battista, 2007, p. 843). In the early grades, geometry instruction should help students make sense of the world around them by building on their intuitive understanding about 2-dimensional (2-D) and 3-dimensional (3-D) shapes and relationships between shapes (Van de Walle et al., 2018a). These experiences are fundamental not only in geometry, but also in other areas of mathematics (Dindyal, 2015) and students' cognitive development (Clements & Sarama, 2021). In addition, research has found that exploring the effects of composing and decomposing shapes is important for students' knowledge of number and arithmetic (e.g., part-part-whole relations and fractions) and executive function processes (Duran et al., 2018; Schmitt et al., 2018).

Measurement and Data

Children apply their understanding of **measurement** concepts in their everyday lives, such as comparing the length of two toys or using vocabulary like "bigger" or "taller." Activities that encourage students to explore measurement attributes, understand the role of the unit, and select and use appropriate units to measure tools, money, and time help them develop a conceptual understanding of the process of measurement (Clements & Sarama, 2021; Schenke et al., 2020; Van de Walle et al., 2018a). Researchers have found that emphasizing these concepts helps students make connections and apply measurement concepts to real-world situations, as well as address common misconceptions (e.g., leaving gaps between units, the need for units to be equal in size, combining units) (Barrett et al., 2017; Sarama et al., 2021).

The expanding use of **data** for decision making in our society has amplified the call for a renewed focus on developing statistically literate citizens (Bargagliotti et al., 2020), even among students in prekindergarten through second grade. To be prepared to work with data, students need to apply statistical reasoning to familiar and everyday situations (English, 2012; Scheaffer & Jacobbe, 2014). Students develop statistical-reasoning skills as they investigate real-world problems and meaningfully interact with data (English, 2013). As young children learn to pose questions, and to collect, classify, organize, represent, and analyze data, they begin to develop a fundamental understanding of statistics (Bush et al., 2014/2015; Van de Walle et al., 2018a).

RESEARCH-BASED RECOMMENDATIONS:

- Integrate **spiral learning** opportunities so students can integrate previous knowledge with newly learned concepts (Bruner, 1960; Harden & Stamper, 1999). Revisit these concepts in increasing complexity.
- **Distribute practice** to maximize student learning, mastery of content, and long-term retention (Schutte et al., 2015; Wiseheart et al., 2019). Utilize technology to supplement instruction, include review quizzes, and integrate activities that provide ongoing opportunities for distributed practice of concepts.
- Promote conceptual understanding of critical mathematics concepts by incorporating the concrete-representational-abstract framework into the design of practice exercises (Agrawal & Morin, 2016; Bouck et al., 2018; Flores, 2010). This framework provides appropriate support and scaffolds that gradually fade as students continue to engage in **meaningful practice** to master the concept or skill.
- Provide a strong focus on **counting** (Clements & Sarama, 2021). Activities should promote fluent verbal counting by encouraging students to learn number words in **sequence** (ordinal numbers), master the backward sequence, and counting on and back from a target number. These activities should also help them understand **one-to-one correspondence** (how each number refers to an item in a set) and make connections to **cardinality** (the last counting word indicates how many objects are in the count). Use technology to help students learn one-to-one correspondence and the cardinality principle (Moyer-Packenham et al., 2015).
- Strengthen students' understanding of **number magnitude** by encouraging them to explore concepts like "more than" and "less than" using concrete objects (e.g., virtual counters), visual representations (e.g., number lines), and symbols (e.g., numerals) (Clements & Sarama, 2021). Students should have opportunities to compare, order, measure, and estimate (Namkung & Fuchs, 2019). Begin with small numbers and gradually increase to larger numbers to help students build a critical understanding of the relative size of numbers (Bay-Williams, 2020; Siegler, 2016).
- Explore patterns in the base-10 system and encourage students to think about **place value** and groups of 10 and 100 (Clements & Sarama, 2021) using concrete and visual representations (e.g., one group of 10 popsicle sticks) (Witzel et al., 2013). Draw attention to the position of a number in multi-digit numbers when discussing place value (e.g., the number 42 is comprised of four 10s and two 1s).
- Provide meaningful practice with **addition and subtraction** to help students develop proficiency with composing and decomposing numbers within five, before transitioning to numbers within 10 and then 20 (Bay-Williams & Kling, 2014; Richardson, 2012). Incorporate real-world contexts and a variety of problem types (e.g., join, separate, part-part-whole, comparison) when teaching these operations (Carpenter et al., 2015). Emphasize strategies (e.g., counting on, properties of operations, break apart to make 10), manipulatives (e.g., counters, base-10 manipulates) and representations (e.g., bar models, number line, five and ten frames) to build conceptual understanding (Clements & Sarama, 2021).
- Nurture students' **algebraic thinking** skills by integrating opportunities for students to engage in arithmetic, understand and apply the fundamental properties of number and operations (Carpenter

et al., 2003; Knuth et al., 2016), develop a relational understanding of equivalence (Blanton et al., 2011; Faulkner et al., 2016), and explore a variety of patterns (Rittle-Johnson et al., 2015).

- Use playful approaches when teaching **geometry** concepts (Clements & Sarama, 2021). Stories help students learn how to use appropriate language when discussing shapes (Nurnberger-Haag, 2016). Activities should encourage students to identify, describe, and analyze a variety of 2-D and 3-D shapes (e.g., rectangular prisms, trapezoids), beyond those traditionally introduced (e.g., circles, triangles). Present a range of materials (e.g., geoboards, virtual manipulatives) for students to compare, classify, compose, and decompose shapes. This helps them attend to shape properties and attributes (e.g., number of sides, sides of equal length) (Van de Walle et al., 2018a).
- Provide real-world scenarios in which students explore **measurement** attributes, nonstandard and standard units, and tools (Clements & Sarama, 2021; NCTM, 2000). Utilize technology to deepen their understanding of measurement, such as a virtual ruler to measure with standard units (Schneke et al., 2020). This helps students understand the concept of measurement and the need for equal length units. Incorporate number lines so students can explore length using a different modality; estimate and compare objects' length, height, and weight; and emphasize precision (e.g., not leaving space between units; correctly aligning an object with a ruler) (Clements & Sarama, 2021; Schenke et al., 2020).
- Allow students to investigate the process of **data** collection, representation, and analysis using real and motivating data sets. Use various graphic representations (e.g., tables, charts, graphs) to organize and display data, reason about the shape of data, and encourage their communication (Biehler et al., 2013; Van de Walle et al., 2018a).

HOW IMAGINE MATH PREK–GRADE 2 INTEGRATES THESE RECOMMENDATIONS:

Spiral learning and **distributed practice** are at the core of the **IM PreK–2** learning environment. Lessons challenge students to explore mathematics concepts over time to help them assimilate new concepts with prior knowledge. For instance, **IM PreK–2** incorporates spiral learning by emphasizing number sense from prekindergarten through second grade. First, lessons focus on developing students' understanding of number sequence, counting, one-to-one correspondence, and cardinality. Then, lessons help students learn how to compose and decompose quantities to five, before moving on to quantities to 10, and finally quantities to 20. This prepares students to compare and order numbers and add and subtract within 10, 20, 100, and 1,000. Distributing practice across grade levels helps students develop an understanding of foundational mathematics content that serves as a backbone for more sophisticated concepts in later grades.

Building a deep conceptual understanding of mathematics takes time. That is why **IM PreK–2** incorporates the concrete-representational-abstract framework into the design of its **meaningful practice** exercises. Intentional scaffolds (e.g., immediate feedback, models) are used to help students make sense of the mathematics concept, but are gradually removed as students demonstrate mastery. For example, **Imagine Math** helps students build meaning of addition and subtraction using concrete representations. In one lesson, students learn to add using concrete objects (e.g., red and yellow flowers) (Figure 20). Students also solve problems contextualized in a story (Figure 21). Students are told that Oliver had seven robots and then gave three robots to Kengji. To illustrate this operation, students select the picture that matches the problem. This checks for understanding and students' ability to match the story to the correct operation. **IM PreK–2** gradually transitions to the use of visual representations, paired with symbolic equations, to help students make connections between the two. For example, students practice finding combinations of 10 using familiar materials (e.g., dice) (Figure 22) and strategies taught in class (e.g., number house) (Figure 23). Finally, students move on to more abstract problems. Students apply their knowledge of the operation and use a variety of strategies to solve word problems (Figure 24) and symbolic equations (Figure 25).

Figure 20. *PK-Lesson 048*

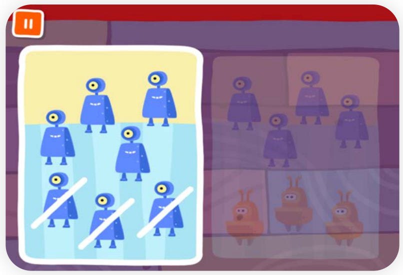


Figure 21. *K-Lesson 020*

Figure 22. *G.1-Lesson 006*

Figure 23. *G.1-Lesson 006*

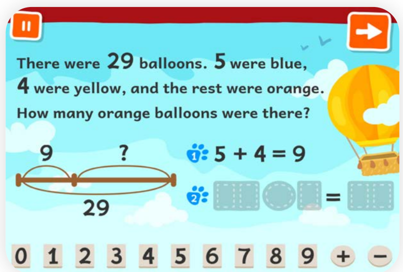


Figure 24. *G.1-Lesson 087*

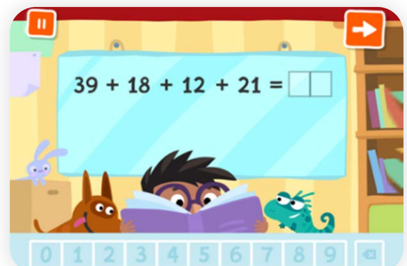


Figure 25. *G.2-Lesson 071*

In recognizing the important role number sense plays in students' overall mathematics achievement, **Imagine Math** provides a strong focus on **counting**, which includes verbal counting, one-to-one correspondence, the cardinality principle, ordinal numbers, and number composition. Figure 26 displays an activity that focuses on the number four. Students practice counting the number of mushrooms and identifying the total number of mushrooms in the group. To promote students' number knowledge, immediate feedback reinforces **one-to-one correspondence** by highlighting each counted mushroom in the set and restating the last counting word to indicate how many total mushrooms are in the set (**cardinality principle**). **Imagine Math** says, "*Let's count together. One, two, three, four. There are four mushrooms. You can do it. Click the number four.*" Another activity within this lesson reinforces ordinal numbers. Students put the leaves in the correct **sequence**, starting with the leaf with the least number of bugs to the greatest number of bugs (1–4) (Figure 27). Developing a strong number sense also includes understanding how numbers are decomposed. **IM PreK–2** lessons encourage students as early as prekindergarten to explore the concept of decomposition (Figure 28) and what it means to decompose the number into two parts (e.g., two and two).

Figure 26. *PK-Lesson 066*

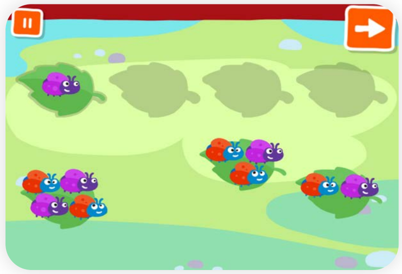


Figure 27. *PK-Lesson 066*

Figure 28. *PK-Lesson 068*

Students learn to **compare numbers** mentally and symbolically, beginning with smaller quantities before moving on to comparing, ordering, and estimating larger quantities. For instance, prekindergarten students learn to compare adjacent numbers to emphasize magnitude understanding (e.g., up to five), as shown in Figure 29. **Imagine Math** provides feedback like "*five is to the right of four, so five is the greater number*," which helps students visualize the placement of each number on their mental number line. In a first-grade lesson (Figure 30), an animation introduces students to the "Hungry Crocodile." Students learn about the meaning of the greater than and less than symbols as the crocodile sings, "*I'm the Hungry Crocodile. And here's my hungry, hungry smile, I'm always facing to the side with the biggest pile. The other pile is way too small, I never face that way at all.*" Students then compare quantities using the crocodile "signs." Later in their pathway, students practice comparing balls with numbers less than 100 (Figure 32). **Imagine Math** reminds students that a "*number line helps organize numbers from lesser to greater*" before prompting them to identify a number less than 45.

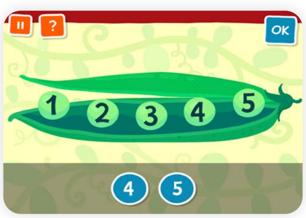


Figure 29. *PK-Lesson 070*

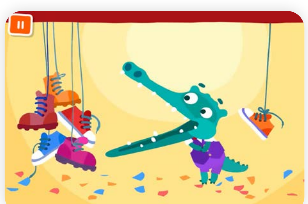


Figure 30. *G.1-Lesson 002*

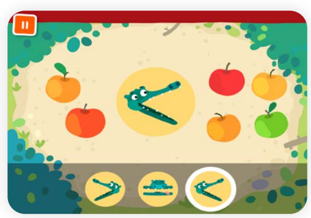


Figure 31. *G.1-Lesson 002*

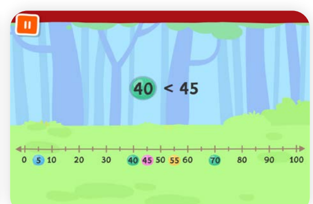


Figure 32. *G.2-Lesson 016*

IM PreK–2 recognizes that many mathematics concepts build on students' knowledge of **place value**. To help students understand the base-10 system, the program integrates concrete and visual representations. In a kindergarten lesson, students are introduced to the number 10 as one group of 10 ones, which are represented as orange sticks (Figure 33). These lessons also use place value tables to provide a visual of how two-digit numbers are decomposed (Figures 34). Lessons model precise language when discussing the value of each digit. In another lesson, students drag place value cards to depict how a number is decomposed (Figure 35). Immediate feedback reminds students that "*One 10 and two ones gives us 12 in total.*" This is important because it helps students learn how to think flexibly about numbers, compare larger numbers (Figure 36), and use decomposition strategies when solving computation problems (e.g., 41 + 16 can be decomposed into 40 + 10 and 1 + 6).

Figure 33. *K-Lesson 021*

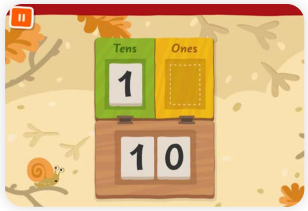


Figure 34. *K-Lesson 021*

Figure 35. *G.1-Lesson 016*

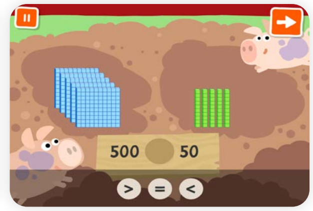


Figure 36. *G.2-Lesson 049*

IM PreK–2 incorporates real-world concepts to which students can relate, visual representations, and models when teaching **addition** and **subtraction**. Within these contexts, **Imagine Math** characters introduce students to important mathematics vocabulary. For instance, in a prekindergarten lesson, Sophia and Ruby introduce students to the concept of subtraction through an engaging real-world animation. As they sit at a kitchen table drinking milk, Sophia and Ruby have a conversation that exposes students to the concept of subtraction. *"Sophia, let's drink together! Less! Decreasing! De-creas-ing. Altogether, decreasing! Dumpling, say decreasing!"* (Figure 27). The next activity in that lesson introduces students to the symbol for subtraction and accompanying vocabulary, which helps build their capacity for engaging in the academic language of mathematics. **Imagine Math** says, *"In math, we use the minus sign to decrease, or subtract. Your turn. Click the minus sign to subtract milk."* (Figure 38). Notice the repetition and connection between everyday language (decreasing) and specific mathematics language (subtract).

Figure 37. *PK-Lesson 044*

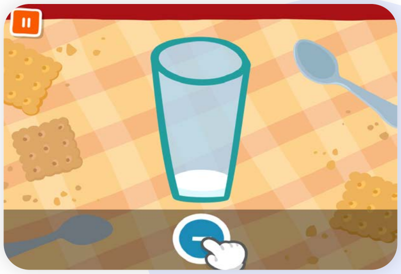


Figure 38. *PK-Lesson 044*

As students progress in their learning pathway, they practice using concrete objects (colored squirrels) to subtract (Figure 39). This helps model the process of subtracting and highlights the connection between the numerals and the illustration. As students' conceptual understanding deepens, they begin using visual representations and models (e.g., segment models) to solve word problems (Figure 40 and Figure 41). While both word problems target subtraction concepts, there are differences in the mathematical structure of these problems. This helps build flexibility in their thinking and use of a variety of strategies when solving.

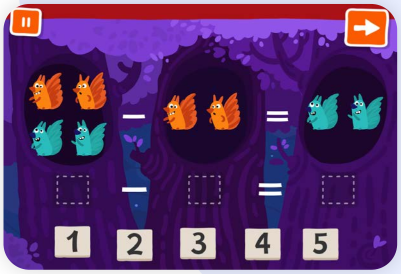


Figure 39. *K-Lesson 012*

Figure 40. *G.1-Lesson 065*

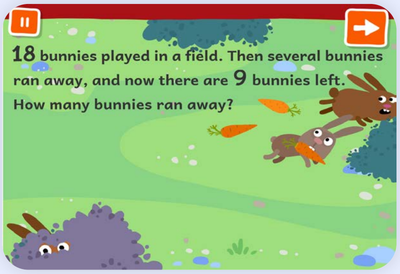


Figure 41. *G.1-Lesson 090*

IM PreK–2 teaches students a variety of strategies to help them become proficient with addition and subtraction, such as counting on from a target number (Figure 42); making 10 to add (Figure 43); skip counting by twos, fives, and 10s (Figure 44); and properties of operations, such as the commutative property of addition (Figure 45). These strategies are also accompanied by models and representations to foster stronger connections to concepts (e.g., number lines, ten frames, part-part-whole models).

Figure 42. *G.1-Lesson 010*

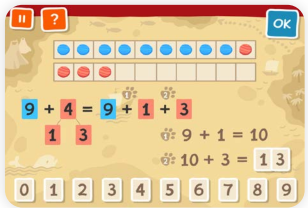


Figure 43. *G.1-Lesson 038*

Figure 44. *G.1-Lesson 124*

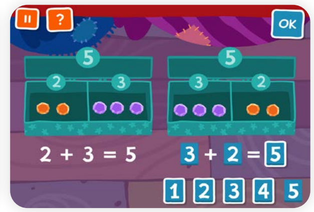


Figure 45. *G.1-Lesson 018*

The **IM PreK–2** learning environment nurtures students' early algebraic thinking by integrating activities that emphasize arithmetic, equivalence, and patterns. For example, students engage in meaningful practice exercises as they solve **arithmetic** problems over the course of the program. Lessons focus on developing facility with addition and subtraction, solving for unknown values, and learning about the fundamental properties of number and operations. For instance, Figures 46, 47, 48, and 49 display examples of age-appropriate arithmetic exercises that students in prekindergarten through second grade solve. Relatedly, students begin solving for unknown values, which are represented by a variety of symbols in **Imagine Math**, such as "x" in Figure 50 and a potion bottle in Figure 51. This is important so students can think flexibly about symbols that can be used to represent an unknown quantity. These types of lessons help lay the foundation for students' future work with variables, equations, and inequalities in algebra.

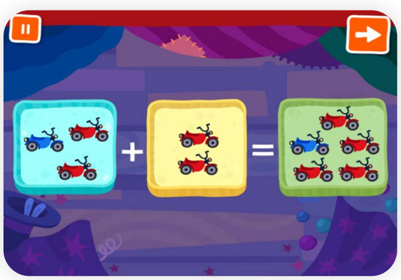


Figure 46. *PK-Lesson 047*

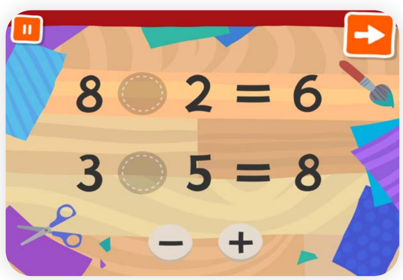


Figure 47. *K-Lesson 042*

Figure 48. *G.1-Lesson 083*

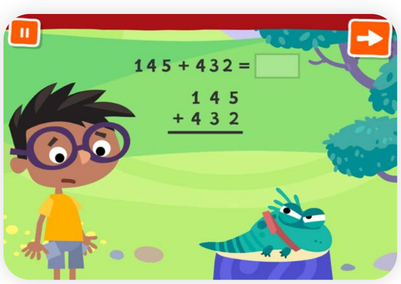


Figure 49. *G.2-Lesson 071*

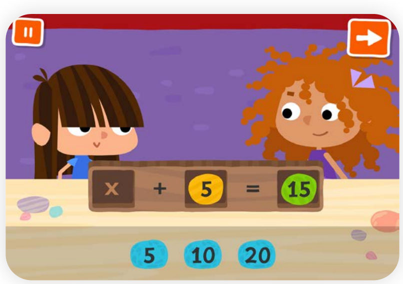


Figure 50. *G.2-Lesson 028*

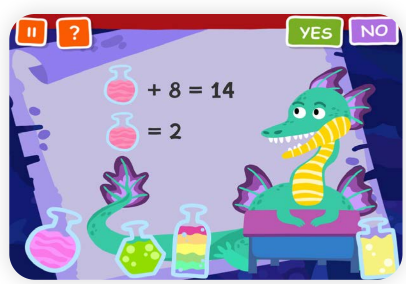


Figure 51. *G.2-Lesson 028*

IM PreK–2 also introduces students to **properties of number and operations**. Lessons encourage students to apply their knowledge of properties and understand why the properties work conceptually. For instance, one lesson provides practice exercises that encourage students to group addends to make adding more efficient (Figure 52). This helps reinforce the associative property, or the idea that changing the grouping of addends does not change the sum. **IM PreK–2** introduces the commutative property by using objects to illustrate how the whole does not change if students switch the order of the parts (Figure 53). This helps them build a conceptual understanding of the property, which they can then apply when analyzing equations like the one shown in Figure 54. In this activity, **Imagine Math** asks students to look at the equation, but not add anything, before providing feedback that reflects on the meaning of the property, *"Switching the order of the addends does not change the sum, so the equation is true."*

Figure 52. *G.1-Lesson 047*

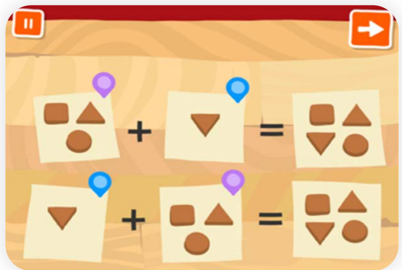


Figure 53. *K-Lesson 022*

Figure 54. *G.1-Lesson 018*

IM PreK–2 recognizes that students need opportunities to develop a relational understanding of the equal sign, rather than understanding it as a symbol denoting an action. The program introduces students to the notion of **equality** through enchanting animations. In one animation, Ruby sings a song about how she and Bingo have the same number of secret treasures (Figure 55). In another, Ruby and her friends demonstrate how the number of cupcakes is not equal to the number of guests (Figure 56). Students are exposed to the symbolic representation of equality and inequality to help them associate meaning with the symbol. In subsequent lessons, students practice identifying whether groups of objects (e.g., goats) are equivalent (Figure 57) and practice comparing quantities using symbols (Figure 58). Later in their learning pathway, students apply their relational understanding of equality by completing number sentences to balance the equation (Figure 59) and generating equivalent expressions using a given set of four numbers (Figure 60).

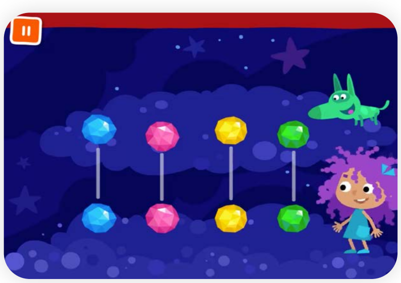


Figure 55. *PK-Lesson 031*

Figure 56. *PK-Lesson 031*

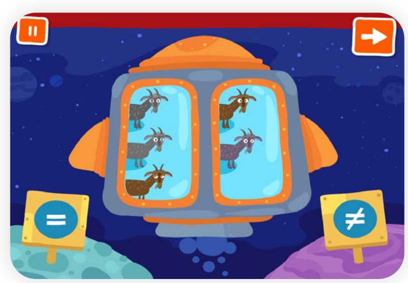


Figure 57. *PK-Lesson 033*

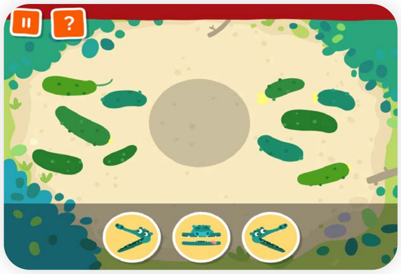


Figure 58. *G.1-Lesson 002*

Figure 59. *G.1-Lesson 018*

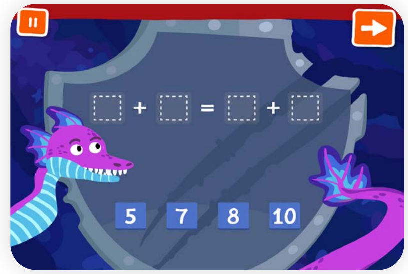


Figure 60. *G.2-Lesson 028*

IM PreK–2 understands that pattern activities are a critical component of early algebra. Lessons integrate **repeating pattern** tasks to help students learn to reason about relationships. For instance, students practice attending to the regularity, repetition, and relationship between the colors of the bugs on each branch (Figure 61) and the colors of the cars (Figure 62) before extending the pattern. **IM PreK–2** also encourages students to explore patterns beyond those depicted using colors. In Figure 63, students explore number patterns by matching equations with their correct model. **Imagine Math** provides feedback that directs students' attention to the pattern (the first addend increases by one and the sum increases by one), which is helpful because this is not always apparent to a young learner.

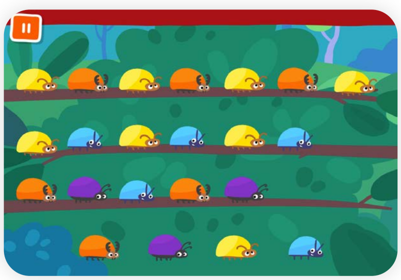


Figure 61. *G.1-Lesson 014*

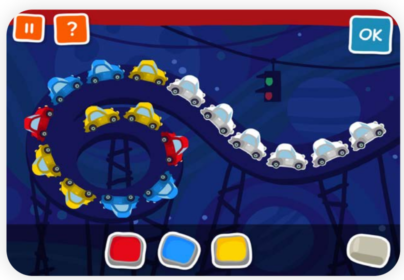


Figure 62. *K-Lesson 076*

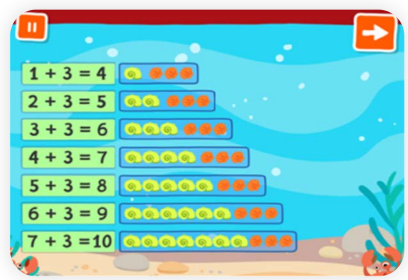


Figure 63. *G.1-Lesson 006*

IM PreK–2 uses playful animations and real-world contexts to teach students **geometry** concepts. These animations are rich with contextualized vocabulary and designed to support students' language development. In one animation, Ruby and friends invite students to come play with them as they put on different masks to become shape heroes (Figure 64). Students practice identifying 2-D shapes to give to the shape heroes (e.g., rectangle, rhombus, trapezoid). In another lesson, students identify, compare, and sort 3-D shapes by dragging the cylinder to the squirrel on the bottom branch (Figure 65). Lessons increase in complexity as students begin to make connections between shapes. In a first-grade lesson, students listen to a story about squirrels who discovered they can stamp flat shapes into the mud. Students determine which 2-D shape the 3-D shape will make when it is stamped into the mud (Figure 66). The program uses precise language when providing feedback, such as *"If we stamp a triangular prism into the mud like this, it will leave an imprint of its base."* **IM PreK–2** also provides opportunities for students to practice manipulating geometric shapes to compose and decompose 2-D shapes (Figure 67). Not only does this deepen students' understanding of shape properties and attributes, but it also provides a foundation for students' experiences with perimeter, area, fractions, and more sophisticated geometry concepts.

Figure 64. *K-Lesson 003*

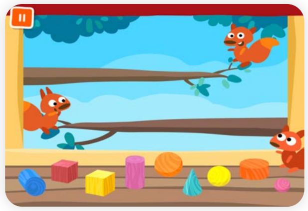


Figure 65. *K-Lesson 082*

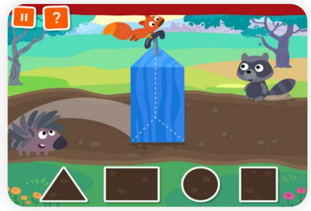


Figure 66. *G.1-Lesson 109*

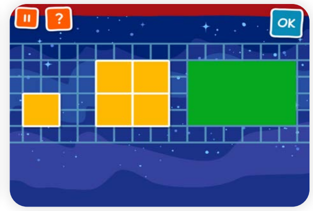


Figure 67. *G.1-Lesson 106*

Songs, animations, and interactive lessons help bring concepts of **measurement** to life. In the **IM PreK–2** learning environment, students learn to measure objects using nonstandard and standard units. In a kindergarten lesson, students listen to a conversation between Ruby and Maya as they talk about measuring the height of bunnies (Figures 68 and 69). Maya asks, *"I wonder exactly how tall I am?"* and Ruby comes up with the idea to measure their height with bunnies, a nonstandard unit. Here, the characters introduce the idea of a unit by saying *"That's right. They'll [bunnies] be our measuring units."*

Figure 68. *K-Lesson 086*

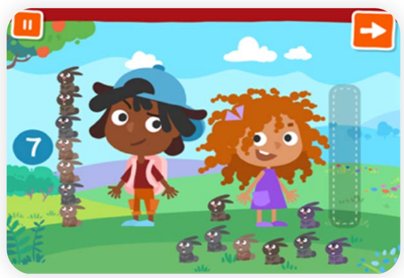


Figure 69. *K-Lesson 086*

Students learn to select appropriate units of measure, estimate before measuring, and use tools to solve problems (e.g., ruler). Students often struggle to measure with precision. **IM PreK–2** uses songs and animations to help students learn how to line up their object, understand what the numbers on the ruler represent, and attend to the unit. For example, characters sing, *"Let's measure a caterpillar! Where's my ruler? Line it up. Line it up. Underneath. On the left. Now, look at the other end. Tick tock. Tick tock. What's the number? What's the unit? It's six centimeters long!"* (Figure 70). Lessons encourage students to determine which tool (e.g., ruler, yardstick, or tape measure) is most appropriate to measure a given object (e.g., fork) and then practice measuring that object (Figure 71). Other activities in this lesson challenge students to measure the same object with two different units (e.g., measuring a ladder in inches and in feet, Figure 72) and compare the lengths of two different objects (Figure 73).

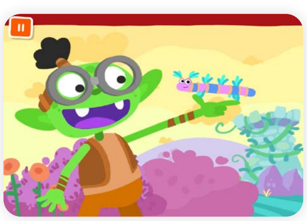


Figure 70. *G.2-Lesson 042*

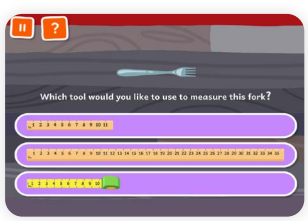


Figure 71. *G.2-Lesson 077*

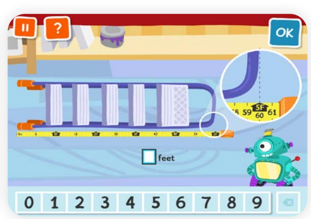


Figure 72. *G.2-Lesson 077*

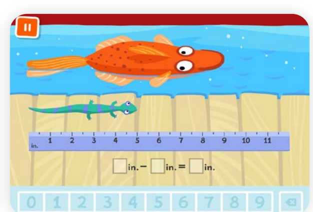


Figure 73. *G.2-Lesson 077*

IM PreK–2 introduces data in interesting and meaningful ways. For example, a series of second-grade lessons expose students to a variety of ways to organize, represent, and analyze **data**. In one lesson, students learn that Oliver found a robot action figure composed of squares, circles, and triangles (Figure 74). This lesson helps students understand what data are represented on a bar graph. Students respond to prompts such as, "*How many squares are on the robot? Use Oliver's bar graph to find out.*" In another lesson, students explore a different graphic representation, a pictograph (Figure 75). Here, students identify the number of cubes, spheres, and cones that were used to create an underwater car. In this lesson, **Imagine Math** draws attention to the importance of a key and how it is used. Without attending to the key, students may respond that there are only three cubes, a common misconception. Students also see how data can be gathered (e.g., pieces of string) and then represented on a line plot (Figures 76 and 77). However, **Imagine Math** recognizes the need for students to go beyond answering questions about a graph to creating one. Figure 78 depicts a lesson in which students explore data collected on Carrot Commander's monthly eating habits. Students learn how to label the Y-axis, create bars to represent the data organized in the table (Figure 79), and answer a series of questions (e.g., "*Did Carrot Commander eat more fruits than vegetables?*").

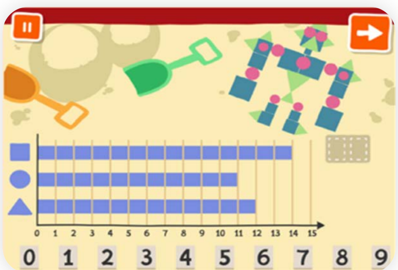


Figure 74. *G.2-Lesson 015*

Figure 75. *G.2-Lesson 036*

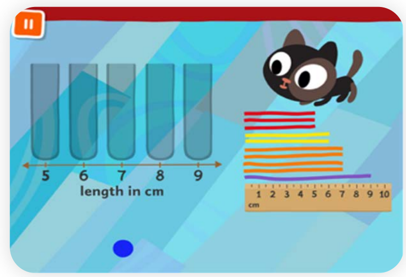


Figure 76. *G.2-Lesson 043*

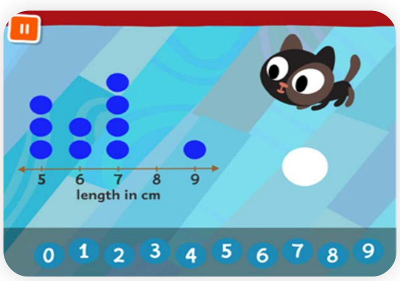


Figure 77. *G.2-Lesson 043*

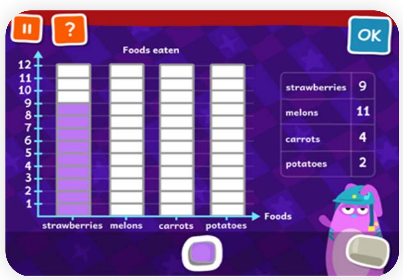


Figure 78. *G.2-Lesson 072*

Figure 79. *G.2-Lesson 072*

Principle 3. Provide a positive digital learning environment that integrates real-world situations to help children grow as mathematicians.

WHAT THE RESEARCH SAYS:

The National Association for the Education of Young Children (NAEYC, 2012, 2020) advocates for the use of technology to optimize children's cognitive, social-emotional, physical, and linguistic development, when used in **developmentally appropriate** ways. Effective **digital learning environments** "are active, hands-on, engaging, and empowering; give children control; provide adaptive scaffolds to help each child progress in skills development at their own pace; and are used as one of the many options to support children's learning" (NAEYC, 2020, p. 13). Multimedia design, digital storytelling, and computer-animated **stories** can be used to cultivate positive and encouraging environments that help shape students' social practices (Starcic et al., 2015), language development (Cooper, 2005; Verhallen et al., 2006), and understanding of mathematics (Islim et al., 2018). Conversational narration and the use of friendly characters who guide learning help children relate to the characters, engage with the content, and make connections to the real world (Clark & Mayer, 2012, 2016). Studies have shown that the use of digital learning programs situated in **real-world contexts** can enhance preschool and elementary children's mathematical thinking, promote positive attitudes and motivation, and help them discover mathematics concepts (Berkowitz et al., 2015; Calder, 2015; Papadakis et al., 2021).

RESEARCH-BASED RECOMMENDATIONS:

- Design a **digital learning environment** that is positive and encouraging (NAEYC, 2020).
  - Incorporate digital storytelling, computer-animated **stories**, and conversational narration to foster cognitive, social-emotional, and linguistic development (Cooper, 2005; NAEYC, 2020; Starcic et al., 2015).
  - Provide **real-world contexts** and utilize friendly characters to encourage relatedness, scaffold learning, and enhance mathematical understanding (Berkowitz et al., 2015; Calder, 2015; Clark & Mayer, 2016; Papadakis et al., 2021).
  - Ensure the digital learning environment is **developmentally appropriate**. This environment should support the individual needs of each child; represent diversity in language, ethnicity, age, and ability; promote active learning and problem solving; build on previously learned content and offer new challenges; and provide immediate feedback (Cooper, 2005). These environments should differentiate learning and encourage children to work at their own pace (Bourbour, 2020; NAEYC, 2020; Papadakis et al., 2021; Sysoev et al., 2017).

HOW IMAGINE MATH PREK–GRADE 2 INTEGRATES THESE RECOMMENDATIONS:

IM PreK–2 cultivates a positive **digital learning environment** that nurtures students' cognitive, social-emotional, and linguistic development. The characters help students grow by modeling positive social interactions and emotions. For instance, when engaging in conversations, the characters are polite, take turns when talking, and help one another solve problems. **IM PreK–2** characters also exude positivity, embrace challenges, and show kindness and empathy. In one animation, Ruby and her friends consider how they can give back to their community after learning about an animal shelter. They come up with an idea to make lemonade to raise money, which they will donate to the shelter (Figure 80). In another example, the dialogue between characters provides unparalleled contextualized learning that promotes academic language. For example, after Oliver told Ruby that he was busy, and therefore unable to play, Ruby offers to help him. The animation seamlessly introduces students to the word "addend" in context before engaging learners in a mathematics activity (Figure 81). Introducing mathematical language early helps students develop communication, reasoning, and effective problem-solving skills that support their language abilities.

Figure 80. *G.2-Lesson 039*

Figure 81. *G.1-Lesson 015*

Imagine Math lessons expound upon engaging, **real-world** storylines to captivate young learners. The **IM PreK–2** environment promotes mathematics learning through lovable characters, songs, and **stories**. Figure 82 displays Ruby, Oliver, and all their **Imagine Math** friends. The characters are intentionally designed to be culturally inclusive and to inspire students to dream big. Science, technology, engineering, and mathematics (STEM)-driven careers are highlighted throughout the narrative; Ruby wants to be an engineer when she grows up, Maya is a scientist in the making, and Sophia is on her way to becoming a doctor. The narration fosters feelings of relatedness and inclusivity between students and **Imagine Math's** friendly characters. For instance, Ruby invites students to go to the pet store with her to look around for a home for her pet fish (Figure 83). In addition, stories are used to activate students' prior knowledge and help them make real-world connections to mathematics content. Many begin with a story to spark interest and curiosity among students before introducing them to the activity (Figure 84). Together, students interact with the virtual characters, explore the virtual world with them, and discover meaning in mathematics.

Figure 82. *IM PreK–2 friends*

Figure 83. *PK-Lesson 092*

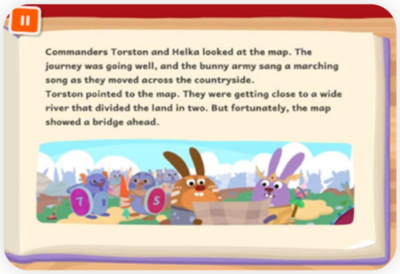


Figure 84. *G.2-Lesson 028*

When a learning environment is **developmentally appropriate**, students grow into competent and confident mathematicians. To accomplish this, **IM PreK–2** introduces students to new mathematical concepts through play with characters they know and love. This approach allows students to master concepts in their own way, at their own pace, and on their own time. Students' learning pathways are adaptive to ensure each child's learning falls within their zone of proximal development. This helps them develop confidence in their own abilities. In addition, immediate feedback, which is warm and encouraging, helps scaffold student learning. For instance, if a student answers incorrectly, feedback includes phrases such as "*You're close. Let's try it again!*" and "*We can do this!*" This type of feedback encourages a positive mathematical mindset, as well as perseverance, by capitalizing on mistakes and repositioning them as opportunities to learn.

Principle 4. Foster a lifelong love of mathematics among early learners by increasing motivation and engagement, and promoting curiosity and confidence.

WHAT THE RESEARCH SAYS:

Research indicates that **motivation** is a strong predictor of student achievement (Lewis et al., 2012; Parker et al., 2014; Skaalvik et al., 2015). With the increased use of technology in early childhood education classrooms, virtual learning environments provide a platform for educators to increase intrinsic and extrinsic motivation among students (Connolly et al., 2012; Papadakis et al., 2021). **Intrinsic motivation** refers to the act of doing something based on internal curiosity, interest, or inherent satisfaction (Filsecker & Hickey, 2014; Ryan & Deci, 2000). Research has found that students who are intrinsically motivated perform at higher levels (Lemos & Verissimo, 2014), are more inclined to persevere when faced with challenges (Huang, 2011), experience greater satisfaction mastering new skills (Elliot & Harackiewicz, 1996), and develop a stronger conceptual understanding of material (Zainuddin et al., 2020). **Extrinsic motivation** reflects one's desire to engage in a behavior that is incentivized or produces an external reward (Moos & Marroquin, 2010). External motives can promote students' willingness to learn (Cameron, 2001; Theodotou, 2014) and behaviors associated with a growth mindset (Mueller & Dweck, 1998; O'Rourke et al., 2014), while verbal rewards can positively influence task completion (Marinak & Gambrell, 2008). Researchers found that focusing on best practices when integrating technology into early childhood education classrooms has led to improvements in engagement, motivation, curiosity, and persistence, especially among young children struggling in mathematics (Larkin, 2013; Moore-Russo et al., 2015; Orlando & Attard, 2016).

RESEARCH-BASED RECOMMENDATIONS:

- Optimize young students' **intrinsic** and **extrinsic motivation** by integrating gamified elements, such as
  - real-world contexts that encourage curiosity and exploration through mastery-oriented quests and challenges (Alsawaier, 2018), as well as relatable characters, engaging storylines, and character dialogue (Bai et al., 2020; Nietfeld et al., 2014; Zainuddin et al., 2020);
  - student choice and interest in order to increase students' feelings of competence and autonomy (Deci & Ryan, 2008; NAEYC, 2012, 2020; Ng, 2018; Nicholson, 2015); and
  - embedded external reward systems to provide continuous feedback (e.g., earned tokens) and opportunities to improve (Filsecker & Hickey, 2014). This can reinforce a growth mindset, or the idea that talents and abilities can be developed through effort and persistence (Mueller & Dweck, 1998; O'Rourke et al., 2014).

HOW IMAGINE MATH PREK–GRADE 2 INTEGRATES THESE RECOMMENDATIONS:

IM PreK–2 increases students' **intrinsic motivation** by incorporating real-world situations that are interesting and relatable to students. Students explore concepts in familiar contexts alongside Ruby and her friends. For instance, Ruby and Oliver explore 2-D and 3-D shapes in a realistic classroom setting. They brainstorm how they could use different shapes to create a city. To excite students, Ruby says to Oliver, *"Look at all of this cool stuff I found! I wonder what we can build with it!"* Oliver responds, *"Let's build a city! And it will be a real city! Everything in it will be 3-dimensional."* These animations help students learn about shape attributes as they listen to conversations taking place between the characters. In Figure 85, Oliver uses a flashlight to make distinctions between the 3-D object (ball) and the 2-D shadow on the wall. Immediately following, students engage in a practice activity in which they determine which real-world objects are in the shape of a sphere (e.g., orange, ball of yarn, clock). For instance, students respond to the prompt, *"What objects are shaped like spheres? Put them between Ruby and Dumpling."* (Figure 86).

Figure 85. *K-Lesson 066*

Figure 86. *K-Lesson 066*

IM PreK–2 provides choices and promotes self-directed learning to **intrinsically motivate** students. Students can choose from a variety of reward environments and skills-based games. For instance, after completing a lesson, students can choose to visit the map and enter targeted review (Treasure Island), listen to songs (Music Hall), play interactive math games (the Fair), or learn more about characters in the Info Center (Figure 87). To foster genuine engagement, **IM PreK–2** also encourages students to set goals. Individual lesson trackers (Figure 88) and achievement certificates (Figure 88), available in English and Spanish, motivate students to take ownership of their learning and success (Figure 89). These elements also help build students' confidence and positive beliefs about their abilities in mathematics.

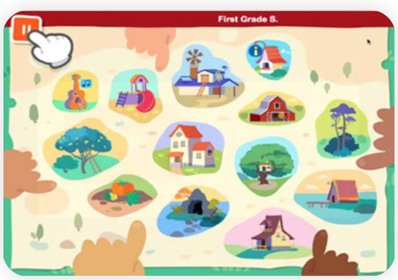


Figure 87. *IM PreK–2 map*

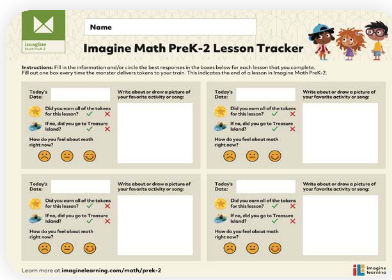


Figure 88. *Lesson tracker*

Figure 89. *Achievement certificate*

IM PreK–2 embeds external reward systems using tokens to **extrinsically motivate** students, celebrate skill acquisition, and reinforce productive behavior (Figure 90). Students earn tokens for completing activities within a lesson with at least 50% accuracy. If students earn all tokens within a lesson, the lesson is considered passed. Students have opportunities to earn back tokens they missed by completing targeted review lessons or visiting Treasure Island to attempt activities that they did not initially pass (Figure 91). This approach helps young learners increase their self-efficacy and perseverance when engaging with challenging activities.

Figure 90. *Tokens*

Figure 91. *Earn back tokens*

Students are encouraged to participate in national contests that target a range of relevant and engaging themes, such as "1, 2, 3, Count With Me!" (Figure 92). This contest motivates students to collect tokens by working through the mathematics lessons and contributing them to their classroom total. This automatically enters their classroom into a random drawing for a e-gift card award. "Winter Math-a-Thon" (Figure 93) challenges students to continue working through and passing mathematics lessons to "score the coolest technology around." Activities such as these have been found to drive student interest and **engagement**.

Figure 92. *1, 2, 3, Count With Me*

Figure 93. *Winter Math-A-Thon*

Principle 5. Differentiate learning by providing informative feedback and adaptive, diagnostic assessments.

WHAT THE RESEARCH SAYS:

Research supports **differentiating instruction** to meet each student's unique learning needs (Hall et al., 2012; Moon, 2016; Subban, 2006; Tomlinson, 2014; Watts-Taffe et al., 2012). Experts in teaching and learning affirm that effective differentiation incorporates informative feedback and adaptive instruction. **Informative feedback** focuses on performance, reinforces correct responses, and provides explanations that help address erroneous thinking or misconceptions (Black & Wiliam, 1998; National Research Council, 2012; Yuan & Kim, 2015). This type of feedback is recognized as one of the most powerful influences on student learning (Hattie & Timperley, 2007). **Adaptive instruction** refers to the instructional content that is modified so learning aligns with students' abilities (Parsons & Vaughn, 2016). Studies have found that adapting instruction throughout a lesson leads to greater learning gains (Aleven et al., 2017). Together, these forms of support identify gaps in understanding, provide remediation to help close those gaps, and advance student understanding (Hattie & Timperley, 2007; Shute, 2008).

Relatedly, researchers argue that **assessments** are an integral part of differentiated instruction and are critical for enhancing students' mathematics learning (Barana et al., 2021; Stacey & Wiliam, 2013; Van der Kleij et al., 2015; Yerushalmy et al., 2017). To garner a holistic view of a young student's mathematical understanding, instruction should draw on both summative and formative assessments (Van de Walle et al., 2018a). Summative assessments measure performance on an outcome measure, whereas formative assessments involve diagnosing student learning needs and adjusting instruction to improve their performance (Schoenfeld, 2015). Formative assessments are considered essential for monitoring student progress, helping teachers make instructional decisions, and improving student achievement (Dalby & Swan, 2019; Faber et al., 2017; Hattie, 2009; NAEYC, 2020; Wang et al., 2019; Wiliam & Leahy, 2015). When teachers use data to identify students' strengths, areas of difficulty, interests, and aptitudes (Kingston & Nash, 2011; Lai & Schildkamp, 2013; Wang et al., 2019; Xie et al., 2019), they can make informed decisions about their own instructional practices and how to best support their students (Faber et al., 2017).

RESEARCH-BASED RECOMMENDATIONS:

- Optimize mathematics learning by providing **differentiated instruction**. Instruction should be adaptive and provide lessons that fall within each student's zone of proximal development (Morgan, 2014; Tomlinson, 2010).
- Incorporate immediate, **informative feedback** to reinforce correct responses, address misconceptions, and promote problem-solving strategies (Belland et al., 2017; Mitrovic et al., 2013; Van der Kleij et al., 2015). This feedback should be clear and purposeful to help students understand and develop effective strategies to engage with the content (Hattie & Timperley, 2007).
- Integrate multiple forms of **assessments** to continuously monitor student progress and growth, understand their thinking, and improve student achievement (Dalby & Swan, 2019; Schoenfeld, 2015). Use information gathered from assessments to make data-driven decisions about instruction and help students attain mastery of grade-level concepts (Faber et al., 2017).

HOW IMAGINE MATH PREK–GRADE 2 INTEGRATES THESE RECOMMENDATIONS:

In **IM PreK–2**, adaptive learning pathways are designed to continuously **differentiate** learning for each student. Figure 94 displays an example of a first-grade student's pathway. Based on the student's Benchmark Test score, the program automatically compacted the content in Lesson B because the student has demonstrated mastery of that content. The student was placed right at their zone of proximal development, in Lesson C. However, if the student had struggled, **IM PreK–2** would have automatically inserted prerequisite lessons throughout their pathway to scaffold learning appropriately. This helps maintain the academic rigor of each lesson and ensure all students can succeed with grade-level content.

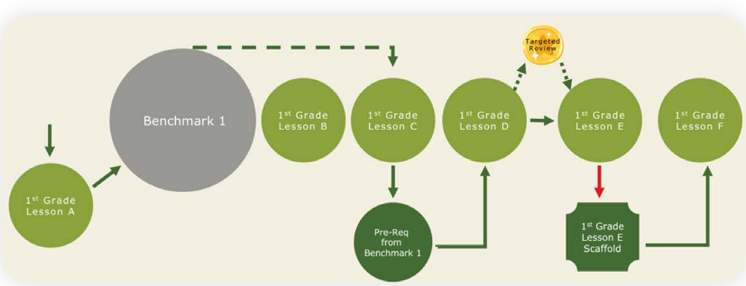


Figure 94. *Student learning pathway*

Imagine Math also differentiates learning within the lessons themselves. As students work through activities in a lesson, they receive ongoing **informative feedback**. For instance, in one lesson, students learn about early fraction concepts, such as partitioning and part-whole relations. The activity begins with a short narrative to engage students: *"Rafa and Zara get to go on a camel ride in the Sahara. The camels will be glad to carry the kids instead of these big packs. Draw a line that divides the rectangle [the big pack] into two halves."* If a student successfully partitions the first rectangle into two halves, they receive praise and feedback in the form of verbal explanations and visual representations. For instance, "*Nice work. Now, can you find another way to divide the rectangle in half?*" If a student struggles, they receive immediate verbal and visual indicators that the answer is incorrect. **IM PreK–2** prioritizes feedback that reinforces a conceptual understanding of these concepts through clear verbal explanations and visual models. For instance, **IM PreK–2** reminds students "*We can also divide the rectangle into two halves like this. Each rectangle is divided into two parts, and the parts are equal in size. Each part is called a half. Since the rectangles are the same size and each part is one half of the whole, all the parts are the same size, even though the parts have different shapes.*"

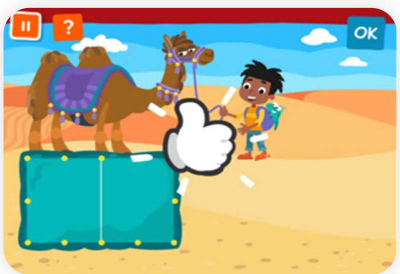


Figure 95. *G.2-Lesson 079*

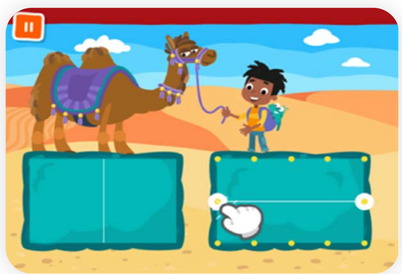


Figure 96. *G.2-Lesson 079*

IM PreK–2 integrates multiple forms of **assessments**. The integrated Benchmark Test series (based on MetaMetrics' Quantile Framework for Mathematics) includes adaptive tests designed to place students and measure student growth and progress.[2] See Figure 97 for an overview. After students explore an introductory lesson to familiarize themselves with the program, they automatically receive Benchmark Test 1. Because this assessment is adaptive, it will look different for every student. This assessment is completed in 1–2 sessions. The result of this Benchmark is a Quantile Measure, performance level (Figure 98), and an instructional grade level. A student's pathway is customized based on a performance level on this Benchmark. Two additional Benchmarks Tests are scheduled over the course of a school year, and will adjust the instructional content a student receives in their pathway as needed. Data from Benchmark Tests provide information on what concepts the student has mastered, as well as any gaps the student needs to close to demonstrate proficiency. In addition, **IM PreK–2** monitors students' progress over the course of the year. Quizzes are embedded in a student's learning pathway to monitor their understanding of related concepts (e.g., addition and subtraction to 20; solving problems involving length).

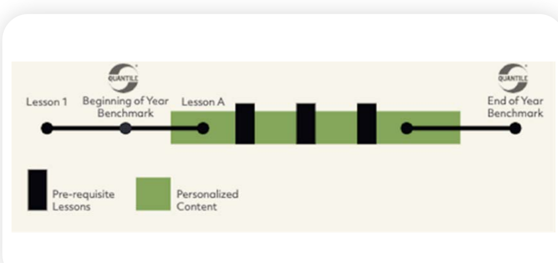


Figure 97. *Multiple forms of assessment*

Imagine Math Performance Levels

GRADE LEVEL/COURSE	PERFORMANCE LEVEL				
	Far Below Basic	Below Basic	Basic	Proficient	Advanced
K	EM400Q & below	EM395Q–EM205Q	EM200Q–EM95Q	EM90Q–410Q	415Q & above
1	EM245Q & below	EM240Q–EM55Q	EM50Q–EM15Q	EM10Q–545Q	550Q & above
2	EM70Q & below	EM65Q–35Q	40Q–185Q	190Q–670Q	675Q & above
3	130Q & below	135Q–235Q	240Q–385Q	390Q–770Q	775Q & above
4	275Q & below	280Q–385Q	390Q–525Q	530Q–910Q	915Q & above
5	340Q & below	345Q–555Q	560Q–685Q	690Q–1005Q	1010Q & above
6	430Q & below	435Q–675Q	680Q–805Q	810Q–1075Q	1080Q & above
7	515Q & below	520Q–795Q	800Q–865Q	870Q–1150Q	1155Q & above
8 / Algebra Readiness	605Q & below	610Q–835Q	840Q–945Q	950Q–1220Q	1225Q & above
Algebra I / HS Math I	680Q & below	685Q–895Q	900Q–1015Q	1020Q–1295Q	1300Q & above
Geometry / HS Math II	730Q & below	735Q–1065Q	1070Q–1155Q	1160Q–1350Q	1355Q & above

Figure 98. *Chart sourced Fall 2021*

Imagine Math recognizes that data should be actionable and used to guide instruction. The program's embedded reporting provides data on students' usage and classroom performance, which help educators identify performance patterns while tracking usage. The Teacher Dashboard enables teachers to manage students, classes, and pathways, as well as view reports. These reports include:

- *Usage Report*—Information about students' use of the program and their performance on Imagine Math lessons
- *Student Activity Report*—Information on how students are using the program (e.g., clicking randomly, repeating the same lesson, or progressing as intended)
- *Student Progress Report*—Information on students' progress in their individualized learning pathways
- *Mastery Report*—Data on students' mastery by standard, which can be used to form instructional groups
- *Benchmark Growth Report*—Allows teachers to track Quantile growth as Benchmark Tests are administered throughout the year

Imagine Math Grade 3–High School Learning Environment

The **Imagine Math Grade 3–High School (IM 3+)** learning environment provides rigorous instruction and incorporates unique motivational elements to prepare students to succeed in mathematics. Students begin by taking a computer-based adaptive Benchmark Test to screen and determine their readiness for mathematics instruction. This enables **Imagine Math** to place each student on a customized learning pathway. Two additional Benchmark Tests are embedded throughout the year, which allow teachers to monitor progress and growth in student achievement. The adaptive algorithms and ongoing assessments powered by the Quantile Framework for Mathematics differentiate instruction so learning is intentionally scaffolded up to grade-level content and beyond. The content is closely aligned with national and state standards to ensure students' experience in **IM 3+** reinforces what they are learning in their core mathematics instruction.

IM 3+ lessons contain a variety of models and representations, incorporate rich academic language, and promote conceptual understanding and procedural fluency. Each lesson builds upon the previous lesson in a student's learning pathway and includes the following activities: Pre-quiz, Warm-up, Guided Learning, Problem Solving, Practice, and Post-quiz (see Figure 99 for an overview). If students struggle with a lesson, **IM 3+** integrates prerequisite lessons that help them develop the necessary background information to master grade-level content. The program also provides intentional scaffolds and immediate feedback to support learning, such as interactive mathematics manipulatives, an interactive glossary, and Math Help tabs. Math Help tabs contain diagrams, modeling, animations, or videos that provide guidance on how to solve the problem. If a student needs more intensive support, they can work with **Imagine Math's** certified, bilingual Live Teacher for one-on-one intervention. This individualized instructional support occurs at the exact moment students need it most. Live Teachers deliver rigorous, differentiated instruction before, during, and after school, on the weekends, and even during school vacations. There is a built-in feedback loop that directly communicates this information to their classroom teacher. To further optimize student learning online and offline, students have access to a vast library of printable resources, most of which are available in English and Spanish. These resources include Journaling Pages (Figure 100), printable worksheets (Figure 101), and STEM Application Tasks (Figure 102). These can be used by students independently or incorporated into small-group and whole-group activities. Instructional games are also included to reinforce specific mathematics concepts and skills (Figures 103 and 104).

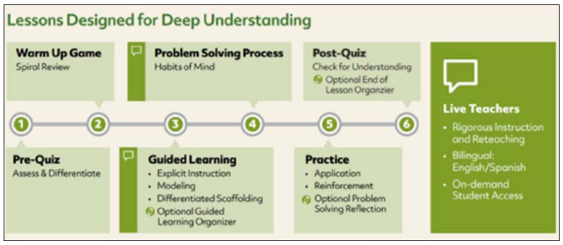


Figure 99. *Overview of lesson*

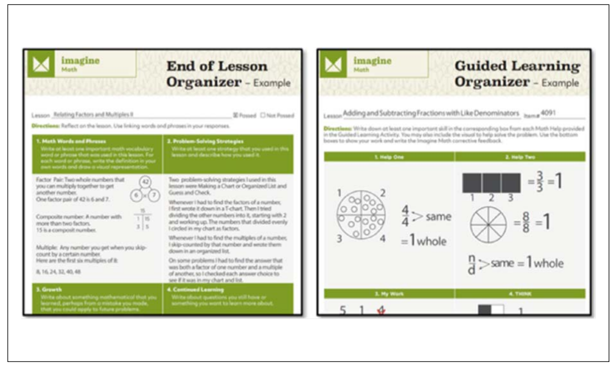


Figure 100. *Journaling Pages*

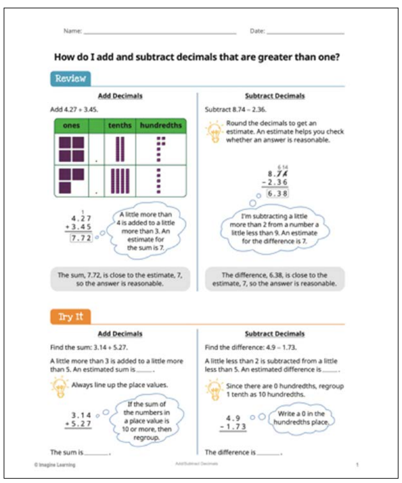


Figure 101. *G.5 Add and subtract decimals*

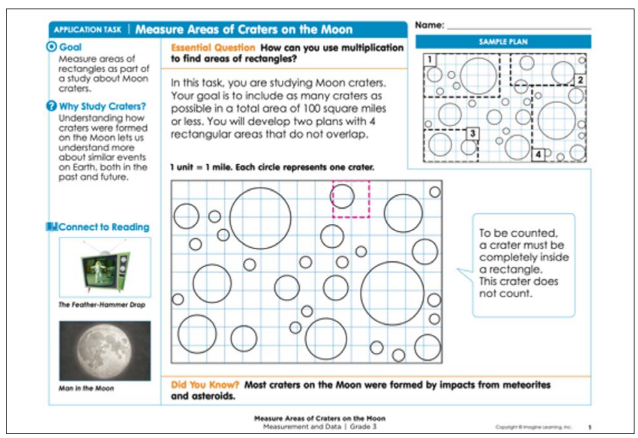


Figure 102. *G.3. Measure Areas of Craters on the Moon*

Figure 103. *Instructional game*

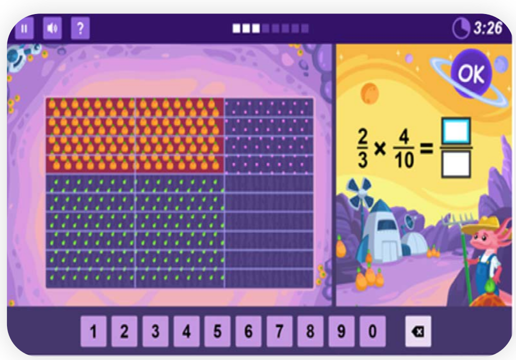


Figure 104. *Instructional game*

In the **Grade 3–High School** learning environment, **Imagine Math** integrates the following research-based principles.

- Provide scaffolded instruction that promotes mastery of grade-level content in number and operations; algebra; geometry; measurement; and data, probability, and statistics.
- Integrate research-based mathematics teaching practices that encourage problem solving, reasoning, and real-world application.
- Promote mathematical discourse to help students develop effective communication skills and a deep understanding of mathematics.
- Ensure all learners receive equitable and accessible mathematics instruction.
- Utilize intrinsic and extrinsic motivational strategies to foster active engagement, collaboration, and perseverance.
- Differentiate instruction by offering informative feedback and adaptive assessments, while providing actionable data to inform mathematics teaching and improve student performance.

Principle 1: Provide scaffolded instruction that promotes mastery of grade-level content in number and operations; algebra; geometry; measurement; and data, probability, and statistics.

NUMBER AND OPERATIONS

WHAT THE RESEARCH SAYS:

Developing a strong number sense, the meaning of whole-number operations, and computational fluency is critical for students' later experiences working with rational numbers, geometry, algebra, and more complex computation problems (Clements & Sarama, 2021; Powell & Fuchs, 2012).

Number Sense

Empirical evidence shows children's early number sense is one of the strongest predictors of later mathematics achievement (Jordan et al., 2010; Nguyen et al., 2016). **Number sense** refers to the ability to understand, represent, and reason flexibly about the relationships between numbers (Green & Towson, 2020). Students follow natural developmental progressions when developing number sense. Research recommends teaching number sense concepts in the following order: subitizing, number recognition, verbal and object counting (including one-to-one correspondence and cardinality), and number magnitude (comparing, ordering, and estimating numbers) (Clements & Sarama, 2021; Powell & Fuchs, 2012; Witzel et al., 2013). However, number sense development does not stop in the early elementary grades. A student's early understanding of number provides a foundation for concepts introduced throughout elementary, middle, and high school (Riccomini & Smith, 2011). These concepts include the structure of the base-10 system (e.g., that the digit in one place represents 10 times as much as the place to its right, and 1/10 of what it represents in the place to its left), place value, and whole-number and rational-number magnitude and computation (Clements & Sarama, 2021; Hickendorff et al., 2019; Siegler et al., 2011; Witzel et al., 2013).

Whole-Number Operations

Facility with whole-number operations (addition, subtraction, multiplication, and division) is critical for learning more complex and advanced mathematics (NCTM, 2000). In the early elementary grades, instruction focuses on students' development of additive reasoning skills, or their understanding of part-whole relations (Vergnaud, 1982). The emphasis is on one unit, where groups are combined successively at one level. Research shows that understanding part-whole relations and mastering single-digit addition and subtraction facts are related to students' proficiency with multi-digit **addition** and **subtraction** (Hickendorff et al. 2019). When students work with larger multi-digit values, it is important that instruction emphasizes the use of reasoning strategies (e.g., decomposing a number to add) to help students understand the misleading concept of regrouping and the process behind the traditional algorithm (Carpenter et al., 2015).

A pivotal shift in learning occurs as instruction transitions to a focus on multiplicative reasoning. This involves reasoning about the relationship between two quantities simultaneously (Vergnaud, 1982). Many scholars argue that understanding multiplicative relations is a necessary precursor to understanding **multiplication**, **division**, ratio, rate, fractions, and algebra (Askew, 2018; Downton & Sullivan, 2017; Ebby et al., 2021; Malola et al., 2020; Siemon et al., 2005). Yet, this transition poses major challenges for students because they are often taught to memorize procedures, rather than the concepts behind the procedures (Dubé & Robinson, 2018). As a result, students have difficulty determining accurate solutions, using effective strategies to solve more complex problems, and explaining their thinking (Baker & Cuevas, 2018). To produce flexible and fluent mathematical thinkers, instruction should capitalize on students' intuitive understanding of equal groups and fair sharing (Bicknell et al., 2016; Empson & Levi, 2011), promote the use of representations, emphasize properties of operations, and encourage reasoning strategies to build fact fluency and conceptual understanding (Schielack, 2010). This prepares students for their work with multi-digit multiplication and division.

Rational-Number Operations

Understanding **rational numbers** is critical for students' work with fraction operations, decimals, percentages, algebra (Bailey et al., 2012), and their overall success in mathematics (Booth et al., 2014; McMullen & Van Hoof, 2020). However, rational number understanding is a source of pervasive difficulty (Tian & Siegler, 2018), especially among students with disabilities (Hunt et al., 2019). Difficulties with fractions often result from the misapplication of whole-number principles (Malone & Fuchs, 2017; Namkung et al., 2018), teachers' inappropriate use of representations, or students' overreliance on rules and algorithms (Flores et al., 2019). Researchers have found that general number-magnitude knowledge plays an important role in helping students learn fraction concepts. Studies show that instruction on fraction magnitude (comparing and ordering fractions, finding equivalencies, placing fractions on a number line) improves students' proficiency with fractions (Bailey et al., 2014; Fuchs et al., 2021). Furthermore, when instruction focuses on helping students make sense of why fractions operations work, students build a deeper understanding of the meaning of the operation (Lamon, 2012; Siegler et al., 2010).

Challenges with rational numbers are further confounded when **decimals** and **percentages** are introduced. Many students hold misconceptions about decimal concepts and continue to overgeneralize whole-number principles and/or procedures (Tian & Siegler, 2018). They also have difficulty translating between notations (e.g., 1/4, 0.25, and 25%) (Muzheve & Capraro, 2012). Because a conceptual understanding of rational numbers "requires understanding the multiple interpretations of rational numbers, skill at translating among the three notations, and knowledge of when each numerical notation is most convenient to us" (Tian & Siegler, 2018, p. 353), students must understand the magnitudes associated with numbers presented in each notation, and develop skills that promote proficiency with arithmetic (Hurst & Cordes, 2015).

RESEARCH-BASED RECOMMENDATIONS:

- Build students' **number sense** by using visuals to help them create a mental representation of the order and magnitude of quantities (e.g., number lines, decimal grids). This helps them learn to compare, order, and estimate whole numbers and rational numbers (Bay-Williams, 2020; Clements & Sarama, 2021; Namkung & Fuchs, 2019; Witzel et al., 2013).
- Strengthen proficiency with **whole-number** and **rational-number** operations by situating problems in real-world contexts that promote flexible problem solving, conceptual understanding, and procedural fluency (Clements & Sarama, 2021; Flores et al., 2019; Van de Walle et al., 2018b; Witzel et al., 2013).
- Teach **multi-digit addition** and **subtraction** using a variety of problem types that encourage students to use manipulatives, models, and reasoning strategies (e.g., decompose numbers by place value to add or subtract) to understand decomposition and regrouping conceptually (Carpenter et al., 2015).
- Help students understand and apply their knowledge of the properties of operations to build **multiplication** and **division** fact fluency (Clements & Sarama, 2021). For instance, use the array model to help students understand the commutative property of multiplication ($2 \times 8 = 16$ and $8 \times 2 = 16$) and decrease the number of facts they are required to learn. Use the distributive property of multiplication ($2 \times 8 = 16$ and $[2 \times 4] + [2 \times 4] = 16$) to help students use known facts to solve unknown facts.
- Teach single-digit **multiplication** and **division** using strategies like repeated addition, equal groups, and arrays to help students make sense of the underlying structure of the operation. Teach multi-digit multiplication and division using strategies area models and partial products to build conceptual understanding before introducing the traditional algorithm (Malola et al., 2020; Milton et al., 2019).
- Promote a conceptual understanding of **fractions** by emphasizing number magnitude. Encourage students to reason when comparing and ordering fractions on a number line (e.g., reason about the size of the parts or estimating using benchmark fractions [0, ½, 1]) (Fuchs et al., 2021).
- Promote proficiency with **fraction operations** by incorporating models (e.g., length models) alongside symbolic equations. Teach conceptual strategies (e.g., finding a common unit) before introducing algorithms (e.g., invert and multiply) (Cramer et al., 2010). Provide explicit feedback on students' misconceptions to address misapplication about whole-number principles (e.g., multiplication does not always lead to a product greater than one of its factors) (Malone & Fuchs, 2017).
- Provide opportunities to fluently translate between **fractions, decimals,** and **percentages**. Draw attention to the equivalency among the three notations and model conversions using a variety of representations (Piper et al., 2010; Tian & Siegler, 2018). Use precise language, such as "two and four tenths," rather than "two point four" (Malone et al., 2017) and emphasize the word "percent" as another way of saying hundredths (Van de Walle et al., 2018b).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ recognizes the importance of helping students think flexibly about numbers. To strengthen students' **number sense**, lessons emphasize number magnitude and incorporate the use of visuals (e.g., number lines, decimal grids) to help them create a mental image of the magnitude of quantities. For instance, a fifth-grade lesson focused on comparing decimals uses different models (e.g., place-value chart, number line, decimal grid) to direct students' attention to number size, magnitude, and reasonableness. Scaffolded feedback encourages students to use a place-value chart to compare the value of each digit and highlights how the value of the digit changes depending on its place-value position. **Imagine Math's** digital manipulatives, or tools, prompt students to use a number line to visually compare 0.062 and 0.62 and each value's proximity to zero and one (Figure 106). Math Helps also help scaffold student understanding by clarifying misconceptions and reminding students that longer decimals are not larger (Figure 107).

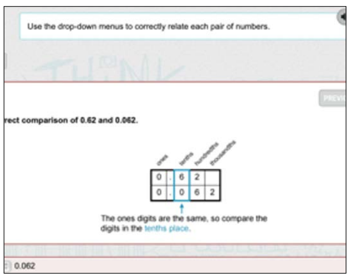


Figure 105. *G.5-Item 93236*

Figure 106. *Digital manipulatives*

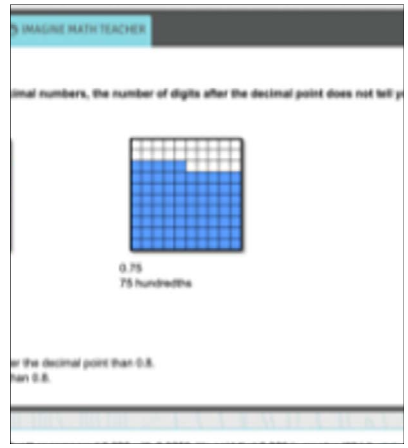


Figure 107. *G.5-Item 93236*

IM 3+ teaches **whole-number** and **rational-number operations** (addition, subtraction, multiplication, division) by incorporating a variety of models and representations to build conceptual understanding. Lessons also emphasize students' use of reasoning strategies to promote flexible problem solving and procedural fluency. For example, in a prerequisite lesson that promotes additive reasoning, **IM 3+** incorporates base-10 blocks to help students understand decomposition and regrouping conceptually when adding and subtracting (Figure 108). Students also solve problems that help them develop efficient mental math strategies (Figure 109). Visual models and audio-recorded video clips scaffold the process for engaging in a mental strategy that could be applied to solve this problem (Figures 110 and 111).

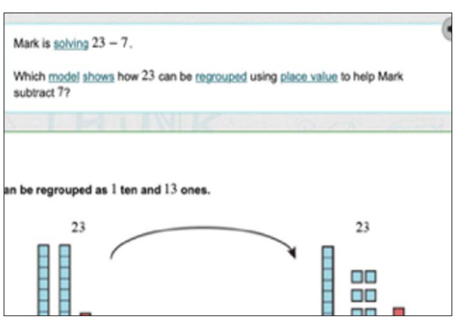


Figure 108. *G.2-Item 4011*

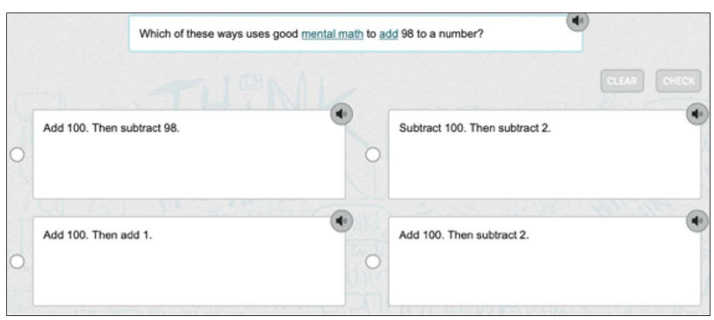


Figure 109. *G.3-Item 4047*

Figure 110. *G.3-Item 4047*

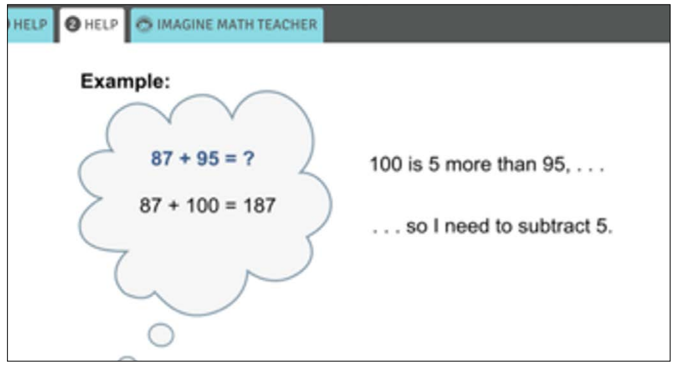


Figure 111. *G.3-Item 4047*

Developing fluency with **multiplication** and **division facts** is strengthened when students understand and apply their knowledge of the properties of operations. **IM 3+** maximizes student learning by minimizing the number of facts to be learned. In Figure 112, this third-grade lesson asks students to consider how Amaya can use her knowledge of a known fact (3 × 7 = 21) to solve an unknown fact (6 × 7). To deepen students' learning, this lesson builds on previously learned strategies (e.g., arrays, number bonds) and integrates feedback that reinforces the meaning of the inverse relationship between multiplication and division.

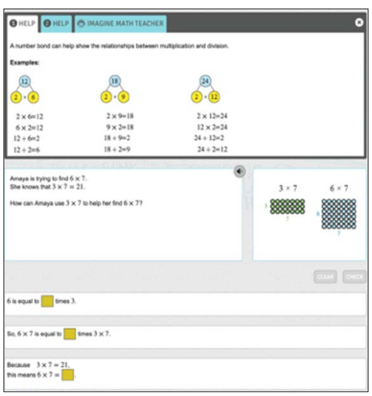


Figure 112. *G.3-Item 41291*

IM 3+ understands the transition from additive to multiplicative reasoning can be challenging. Therefore, **IM 3+** strategically sequences lessons to support students' conceptual understanding of the operations. In this learning environment, the program introduces **division** using real-world problems and concrete representations (Figure 113). Lessons incorporate the use of "groups of" language to help students attend to the important relationship between the number of groups and number of objects in each group. **IM 3+** gradually incorporates symbolic expressions alongside representations to facilitate a deeper understanding of division (Figure 114). **IM 3+** recognizes the difficulty associated with interpreting remainders; therefore, scaffolded feedback uses visuals (e.g., arrays) to help students make sense of the equation (17 ÷ 3 = 5 R2) and the true meaning of the remainder "R2" (Figure 115). As students demonstrate proficiency, they solve more complex multi-digit problems and make stronger connections between the use of models (e.g., area models) and expressions (Figure 116).

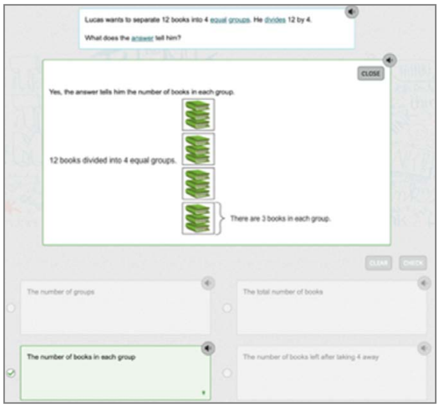


Figure 113. *G.3-Item 4603*

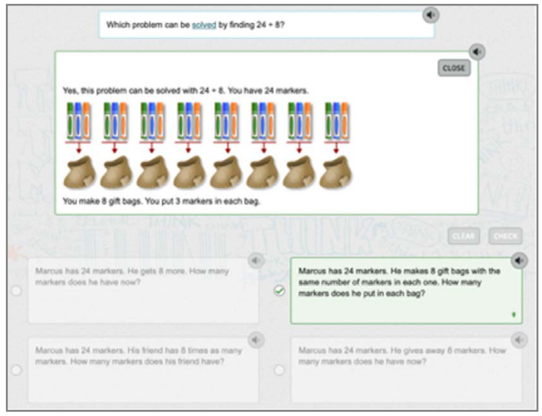


Figure 114. *G.3-Item 4050*

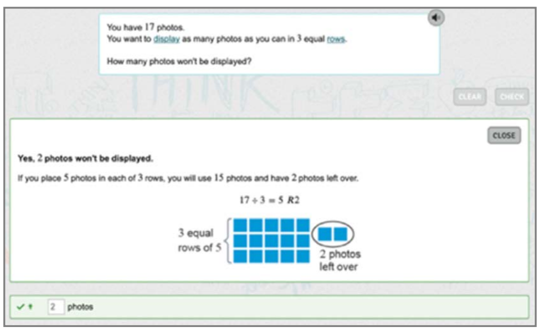


Figure 115. *G.3-34588*

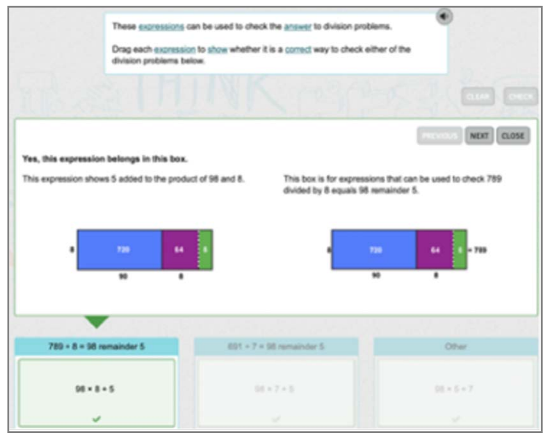


Figure 116. *G.3-Item 94122*

IM 3+ helps students develop a strong understanding of rational numbers by emphasizing fundamental concepts like **fractions** as numbers, unit fractions, partitioning, iterating, and part-whole relations. For instance, number lines are used to illustrate the relationship between equal-sized parts and the whole (Figure 117). Feedback models precision when discussing this relationship (e.g., *"The fraction ⅓ means one part of three equal parts," rather than "one out of three"*). This is important because students' incorrect use of language can lead to misconceptions about fractions. **IM 3+** also focuses on building students' understanding of fraction magnitude. When students explore this concept, scaffolds and feedback help students compare and order fractions using models and representations (e.g., area models, Figure 118; fraction strips, Figure 119), reasoning strategies (e.g., the size of the parts, Figure 120), and estimation strategies (e.g., benchmark fractions like 0, ½, 1, Figure 121).

Figure 117. *G.3-Item 94031*

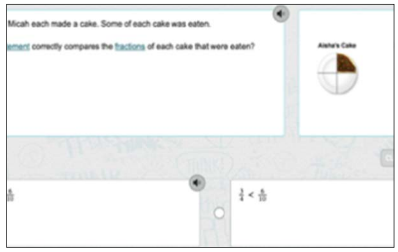


Figure 118. *G.4-Item 30440*

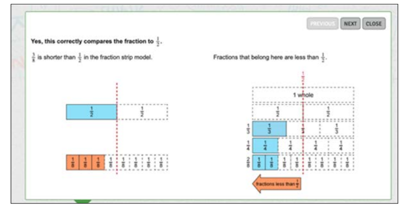


Figure 119. *G.4-Item 30440*

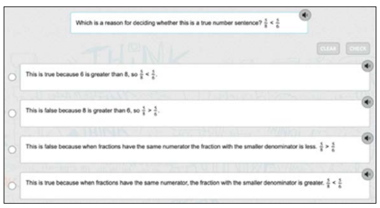


Figure 120. *G.3-Item 6989*

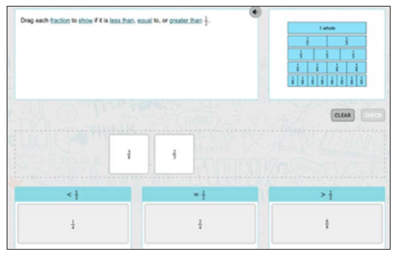


Figure 121. *G.4-Item 30440*

To promote proficiency with fraction operations, **IM 3+** utilizes word problems and visual models to promote students' use of conceptual strategies. For instance, lessons draw explicit attention to the need for a common unit when **adding** and **subtracting fractions** (Figure 122). Scaffolded feedback recommends finding a common denominator, or the same-sized whole, when adding the fractional parts $3\frac{1}{5} + 2\frac{2}{3}$. In another lesson, students use models and estimation strategies to add fractions more easily (Figure 123). Feedback models the problem using a part-whole diagram and questions that prompt students to predict, or estimate, whether the amount will be greater than or less than one cup.

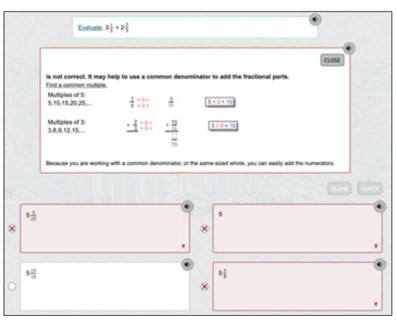


Figure 122. *G.5-Item 4233*

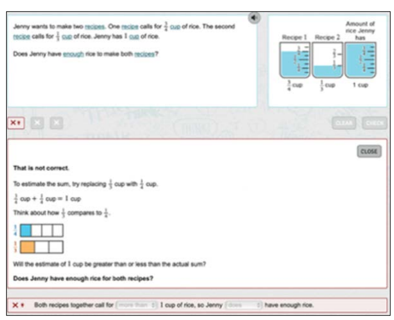


Figure 123. *G.5-Item 34634*

While adding and subtracting fractions is typically easier for students, **multiplying** and **dividing fractions** is often a source of struggle. **IM 3+** encourages students to make sense of the problem conceptually. Lessons promote students' use of visual models to build meaning of the operation and make connections across representations (e.g., real-world contexts, fractions, models, expressions) (Figures 124 and 125). In Figure 125, students match the word problem with the correct division expression. To solidify understanding, immediate feedback models the procedure using a part-whole diagram in order to represents how many $\frac{1}{2}$-cup servings are in a $2\frac{1}{2}$-cup container ($2\frac{1}{2} \div \frac{1}{2}$), rather than prompting them to solve the problem procedurally.

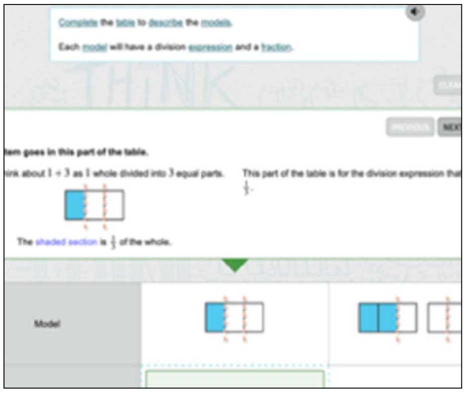


Figure 124. *G.5-Item 33646*

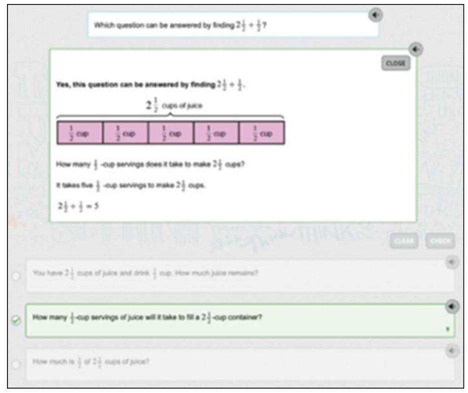


Figure 125. *G.6-Item 4330*

Students also learn to translate across **fractions, decimals,** and **percentage** notation. In a seventh-grade lesson on equivalence, students explore how these notations are different representations of rational numbers. Students solve problems that ask them to determine which sale would give them the greatest discount (Figure 126) and practice converting between forms (Figure 127). Feedback provides examples of equivalent notations and reinforces students' understanding of critical vocabulary words (e.g., "percent" means "out of 100," so 25% means 25/100 or 0.25) (Figure 128).

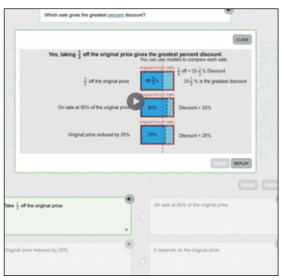


Figure 126. *G.7-Item 4339*

Figure 127. *G.7-Item 4339*

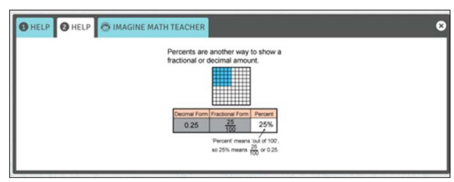


Figure 128. *PK.7-Item 4339*

ALGEBRA

WHAT THE RESEARCH SAYS:

Algebra is commonly referred to as the linchpin to school and career success (Knuth et al., 2016). The importance of developing algebraic thinking among students has been well documented in the literature (Beatty et al., 2013; Blanton et al., 2019; Carraher & Schliemann, 2018). Yet, many students hold several misconceptions about algebra (Welder, 2012) and existing curricula have often ill prepared students to reason in ways that are consistent with higher-level algebra concepts (Knuth et al., 2016). In these instances, algebra can serve as a "gatekeeper" that deters the continued study of mathematics and limits "access to college majors and careers" (Carpenter et al., 2003, p. 6). To improve students' algebraic thinking, exposure to a range of concepts across grade levels is needed (Blanton et al., 2019; Kaput, 2008; Kieran et al., 2016).

- **Generalized Arithmetic, Properties,** and **Variables.** Often, students struggle with arithmetic because they lack an understanding of mathematical structure of operations (Mason, 2016), do not see the connection between properties used in arithmetic and those used in algebra (Carpenter et al., 2003), and confuse a variable for a label (Russel et al., 2011). Experts recommend students learn to reason about the structure of an expression, apply the properties of operations, and use variables to represent unknown quantities (Banerjee & Subramaniam, 2012; Blanton et al., 2017; Kieran, 2018; Pang & Kim, 2018).

- **Equivalence.** Studies show many students lack a relational understanding of equivalence (Carpenter et al., 2003; Knuth et al., 2016; Stephens et al., 2013). Using language like "the same as" when describing equality is problematic because two equal values are not "the same" (Faulkner et al., 2016). Many students also regard the equal sign as a symbol of action, rather than a symbol denoting the relationship between two quantities (Powell et al., 2020). For instance, students who answer $5 + 4 = \_\_ + 7$ is 9 likely do not attend to the relationship between the quantities (McNeil et al., 2017). Studies show that students who interpret the equal sign as a symbol of action perform lower on algebra tasks, with more profound negative consequence across grade levels (Byrd et al., 2015).
- **Patterns, Relationships,** and **Functions.** Research recommends that instruction provide students with opportunities to explore patterns and functions (NCTM, 2000). Scholars have found that geometric and numeric patterns are a powerful means for stimulating children's algebraic thinking and predisposing them to think about functional relationships (Beatty et al., 2013; Blanton et al., 2015; Canadas et al., 2016; Vanluydt et al., 2021; West, 2021). Functional thinking involves "generalizing relationships between co-varying quantities…through natural language, variable notation, drawings, tables, and graphs" (Blanton et al., 2018, p. 33). Integrating these opportunities helps students understand, justify, and generalize quantitative relationships across multiple and more sophisticated concepts (Ellis, 2011).
- **Generalizing.** Generalizing is often described as the heart of algebraic thinking (Hashemi et al., 2013; Kieran, 2018; Mason et al., 2010). Students' generalizations can include words, pictures, diagrams, graphs, and symbolic notation, among others. While learning to generalize can be challenging, studies show when students are taught to generalize, they have made statistically significant gains on tasks focused on generalizing, representing mathematical relationships, and structure (Blanton et al., 2019).

RESEARCH-BASED RECOMMENDATIONS:

- Provide opportunities for students to explore **properties** of operations (e.g., distributive property of multiplication) and generate equivalent expressions to demonstrate their understanding of that property (Knuth et al., 2016). This helps students transition from words to variables ($a + b = b + a$) (Carpenter et al., 2003). Students should solve expressions with one and more than one variable to explore how a **variable** can represent an unknown quantity or quantities that vary (Stephens et al., 2015).
- Promote a relational understanding of **equivalence** to contradict the misconception that the equal sign acts as a symbol of action rather than as a relational symbol indicating both sides of the equation have the "the same value" (Blanton et al., 2011; Carpenter et al., 2003; Faulkner et al., 2016; Powell et al., 2020).
- Leverage **patterns** to support students' algebraic thinking skills (Blanton et al., 2015; West, 2021). Activities should incorporate a variety of representations (e.g., geometric shapes, pictures, numbers), encourage reasoning about additive and multiplicative relationships, and promote explanations and justifications as students learn to **generalize** the pattern rule.

- Develop students' understanding of **functional relationships** by engaging them in real-world problems (Ellis, 2011) that encourage reasoning about the co-varying relationship between quantities and representing that relationship using language, models, and algebraic notation (e.g., $2x + 2 = y$) (Knuth et al., 2016).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

Recognizing the importance of generalized arithmetic, **IM 3+** lessons are intentionally designed to support students' **arithmetic** skills. Lessons encourage students to make sense of what variables represent conceptually, rather than emphasizing arcane rules for manipulating symbols (Figure 129). For instance, given a word problem, students identify "what g represents" or the unknown value for which they are solving. This helps students focus on what is happening in the problem mathematically. They also apply their knowledge of the **properties** of number and operations to solve equations with one **variable** (Figure 130). In this example, informative feedback clarifies how to apply the distributive property to simplify the equation and models how to correctly manipulate variables to ensure the equation is equivalent. This feedback provides the basis for symbolic manipulation, while also helping students recognize relationships between procedures and concepts in arithmetic and algebra.

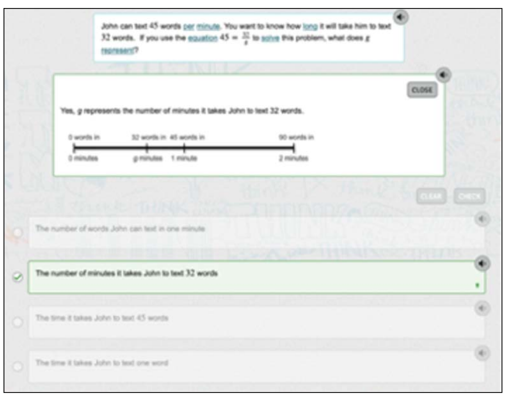


Figure 129. *G.6-Item 8019*

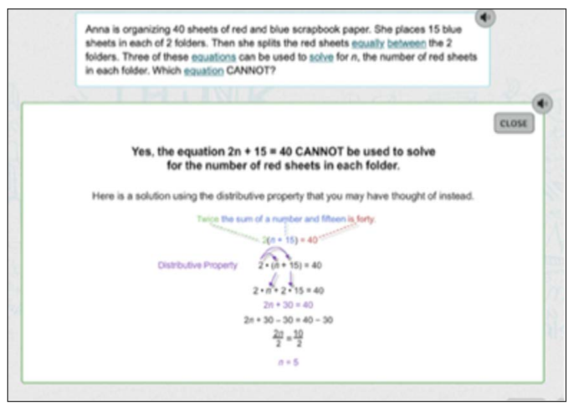


Figure 130. *G.6-Item 4453*

As students progress in their pathway, they write and graph linear equations with two or more **variables**. Students use ordered pairs derived from tables, algebraic rules, or descriptions to graph linear functions. In one eighth-grade lesson, students analyze components of the algebraic expressions (e.g., coefficients, constants) to determine the function (Figures 131 and 132). Multiple representations are used to help build students' conceptual understanding of what occurs in the problem (e.g., context, tables, equation, graphs). If students struggle to complete the statements about the system of equations displayed in Figure 131, the Math Helps provide intentional scaffolds that model two different ways to solve the problem (substitution method, graphing) (Figures 133 and 134, respectively).

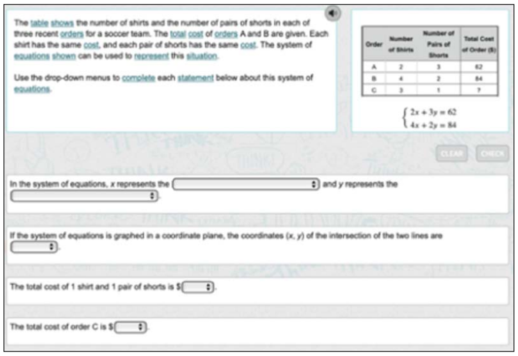


Figure 131. *G.8-Item 98238*

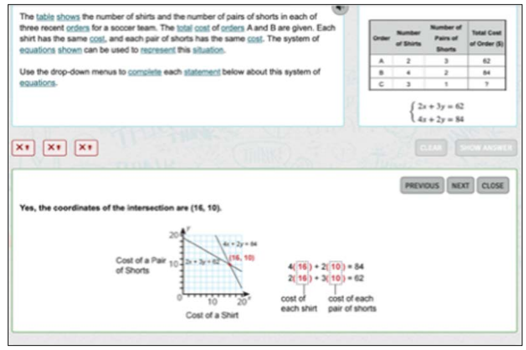


Figure 132. *G.8-Item 98238*

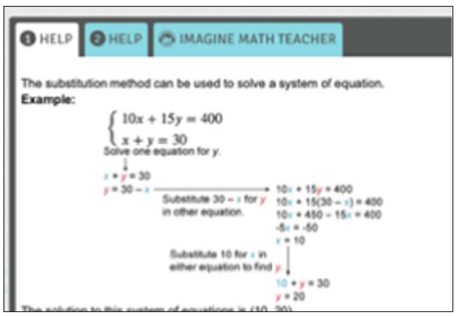


Figure 134. *G.8-Item 98238*

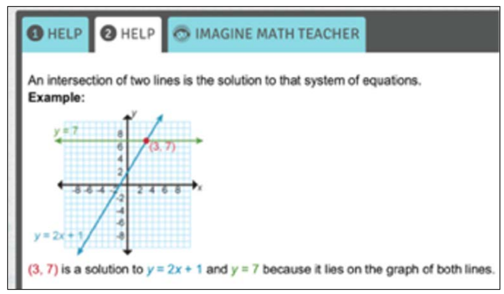


Figure 135. *G.8-Item 98238*

IM 3+ promotes a relational understanding of **equivalence** to combat the misconception that the equal sign acts as a symbol of action. This seventh-grade problem reinforces the idea that the equal sign is a relational symbol indicating both sides of the equation are equivalent (Figure 136). Scaffolded support helps students learn the process for correctly combining like terms to determine if the two expressions are equivalent.

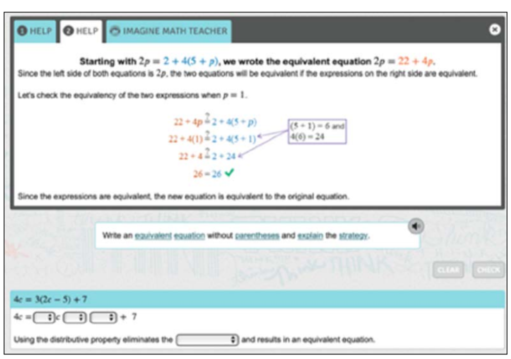


Figure 136. *G.7-Item 34706*

Students' exploration of **patterns, relationships,** and **functions** helps build their understanding of mathematical relationships. In one fourth-grade lesson, students engage in early functional thinking as they investigate the relationship between two covarying quantities and **generalize** the pattern rule (e.g., "double and then add 2") (Figure 137). Tables are used to help students organize their thinking and draw attention to the relationship between the quantities. Immediate feedback models how to analyze the pattern and uses color coding to reinforce other important mathematics concepts (e.g., multiples, place value) (Figure 138).

This lesson, among others (Figure 139), has important implications for students' later work with functions and generalizations.

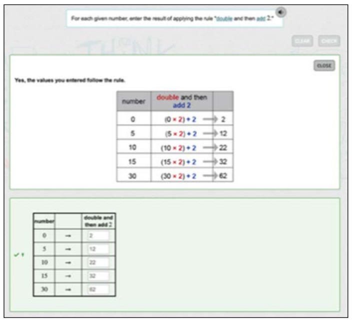


Figure 137. *G.4-Item 4267*

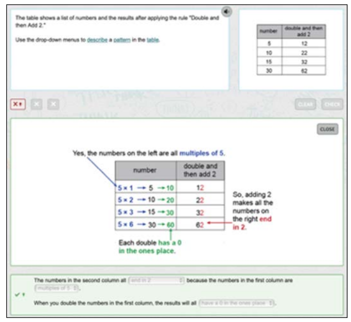


Figure 138. *G.4-Item 4267*

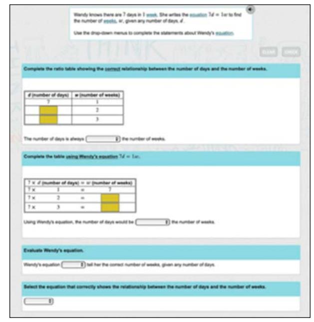


Figure 139. *G.6-Item 98021*

GEOMETRY

WHAT THE RESEARCH SAYS:

Spatial thinking involves recognizing and manipulating properties of shapes and the relationship among them (Mulligan, 2015). Spatial thinking includes spatial visualization (the mental ability to operate, rotate, and turn an object) and spatial orientation (the ability to orient an object or shape within a given spatial location) (Chao & Liu, 2017). Research has found that spatial thinking is an important part of mathematical thinking, and studies reveal strong relationships between students' spatial thinking and overall mathematics performance (Lowrie et al., 2019; Mix et al., 2016; Rittle-Johnson et al., 2019; Wei et al., 2012).

Geometry is commonly associated with spatial thinking and is described as "a network of concepts, ways of reasoning, and representation systems" that challenges students to explore and analyze shapes and space (Battista, 2007, p. 843). National standards recommend all students explore characteristics of 2-D and 3-D shapes; construct arguments about geometric relationships; determine locations using coordinate systems to analyze geometric situations; and apply transformations (NCTM, 2000). Students' early experiences with geometry build a foundation for the depth and sophistication of their thinking in middle and high school.

The van Hiele Model of **Geometric Thinking** catalogues students' progressive understanding of geometric and spatial ideas (van Hiele, 1986). The hierarchy of these five levels is not characterized by age; rather, students' progression is dependent upon their learning experiences (van Hiele, 1999).

- Level 1 (Visualization)—identifies geometric shapes, but does not focus on properties or attributes
- Level 2 (Analysis)—recognizes shapes have different properties, and can identify shapes by that property, but does not recognize the relationship between properties
- Level 3 (Informal Deduction)—recognizes and describes the relationships between objects and shapes, and engages in "if...then" reasoning
- Level 4 (Formal Deductive)—constructs proofs, analyzes informal arguments and the structure of a system, and begins to establish geometric truth based on logic
- Level 5 (Rigor)—understands abstract geometry and sees the "construction" of geometric systems

RESEARCH-BASED RECOMMENDATIONS:

- Bolster students' **spatial thinking** by integrating activities that encourage students to think about, manipulate, and transform shapes mentally and physically (Clements & Sarama, 2014; Ontario Ministry of Education, 2014), as well as engage in mathematical modeling (Hodgson & Riley, 2001).
- Integrate **geometry** activities that encourage students to analyze a variety of 2-D and 3-D shapes (e.g., prisms, different types of triangles), beyond those traditionally introduced (e.g., circles, squares) (Clements & Sarama, 2021). Present a range of materials and tools (e.g., pattern blocks, geoboards, virtual manipulatives) for students to use to compare and classify examples and non-examples. This helps students attend to shape properties and attributes (number of sides, sides of equal length).
- Design lessons that draw on van Hiele's Model of Geometric Thinking to support students' progression through the five levels of **geometric thinking** (Breyfogle & Lynch, 2010).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ promotes **spatial thinking** by providing lessons that encourage students to manipulate and mentally transform 2-D and 3-D shapes. For instance, Figure 140 depicts a problem in which students mentally transform 2-D figures with reflections, rotations, and translations to build meaning of congruence.

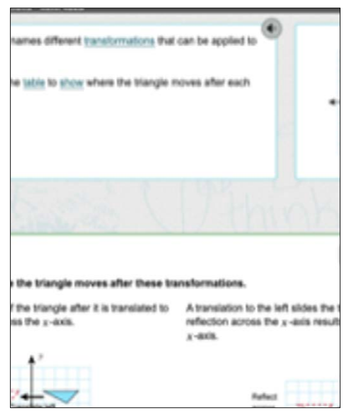


Figure 140. *G.8-Item 4823*

Students also strengthen their **spatial thinking** skills as they explore how space is organized, labeled, and described. **IM 3+** recognizes that students' early experiences using coordinates to navigate and describe locations help them understand the Cartesian coordinate system, a concept directly related to algebra. Lessons provide students with opportunities to relate the order of coordinates to their location on the coordinate plane. Immediate feedback reinforces key mathematical terms, such as the numbers in the ordered pair, the origin, and how far and in which direction to travel from the origin (Figure 141). This helps students reflect on their own understanding while also equipping them to acquire background knowledge needed to solve more sophisticated problems involving area, surface area, and volume in the coordinate plane (Figure 142).

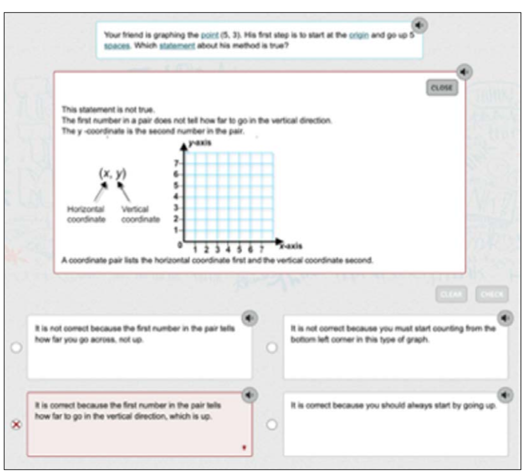


Figure 141. *G.5-Lesson 3289*

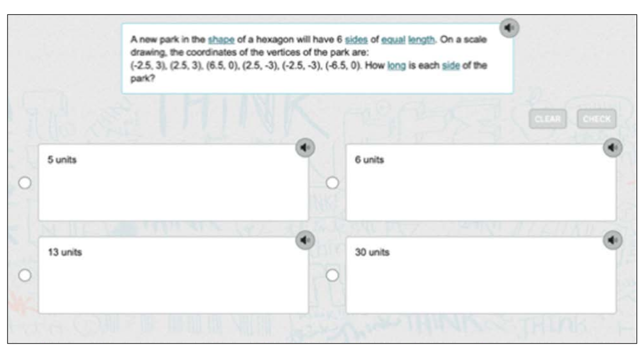


Figure 142. *G.6-Item 4257*

Students investigate a variety of **geometry** concepts throughout the program, such as 2-D and 3-D shape properties and attributes. They also learn how to categorize, classify, and compare examples and non-examples. For instance, in one third-grade lesson, students practice identifying faces, edges, and vertices of complex 3-D figures (Figure 143). Feedback explicitly helps address common misconceptions by modeling clear and concise mathematical language (e.g., *"These are the rectangular faces. Remember, squares are a type of rectangle."*) (Figures 144 and 145). According to the van Hiele Model of **Geometric Thinking**, this lesson would be categorized as Level 2 (Analysis) because of its focus on recognizing and describing different shape properties. In more advanced geometry lessons (Figures 146 and 147), students reason at increasingly sophisticated levels as they solve problems that focus on congruence in terms of rigid motion to prove geometric theorems about lines and angles. Here, this geometry lesson would be categorized as Level 4 (Informal Deductions) because it challenges students to construct proofs and establish geometric truth based on logic.

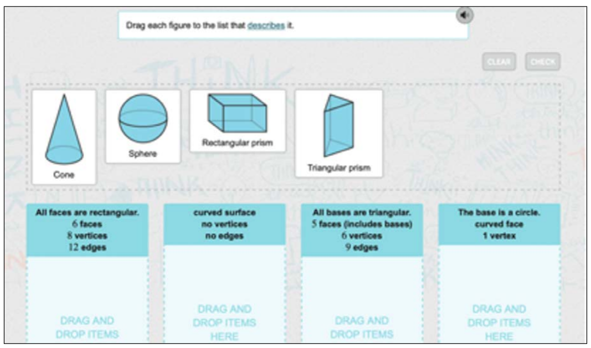


Figure 143. *G.3-Item 34555*

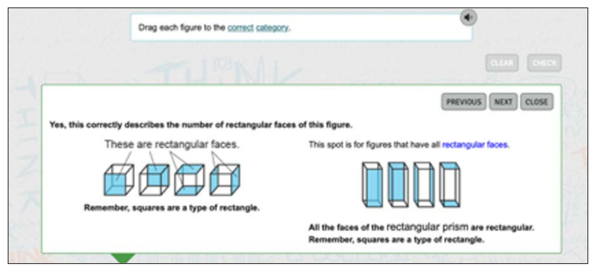


Figure 144. *G.3-Item 34555*

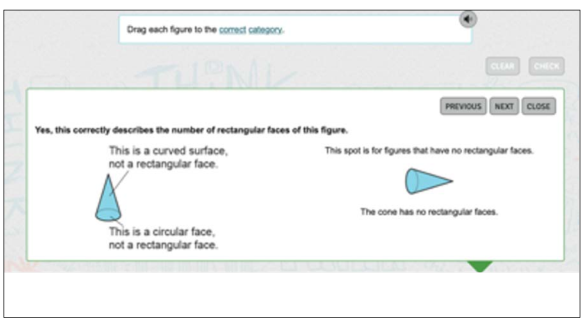


Figure 145. *G.3-Item 34555*

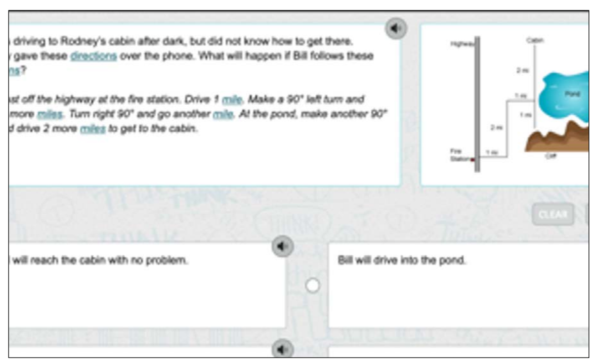


Figure 146. *Geometry-Item 50596*

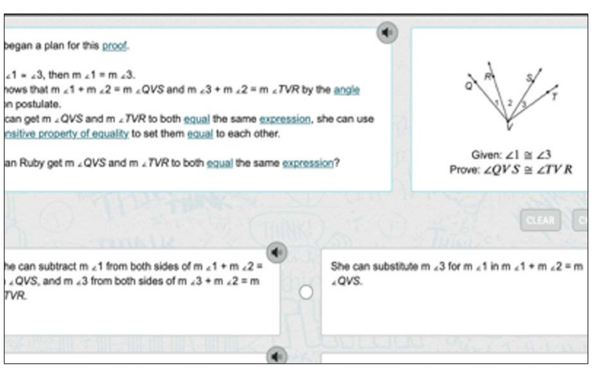


Figure 147. *Geometry-Item 50596*

MEASUREMENT

WHAT THE RESEARCH SAYS:

Measurement is defined as “the process of assigning a number to a magnitude of some attribute—a continuous quantity—of an object relative to a unit” (Clements & Sarama, 2021, p. 247). Developing a conceptual understanding of the meaning and process of measurement is important for learning to make connections and apply measurement concepts to real-world situations. This includes learning to measure in customary and metric systems, measurement conversions, and selecting and using appropriate units and tools, as well as concepts like length, perimeter, area, surface area, and volume (NCTM, 2000). Not only does measurement bridge domains of number and geometry (Clements et al., 2017), but it also relates to students’ spatial thinking (Reinhold et al., 2020) and more advanced concepts like fractions and decimals (Brendefur et al., 2013). However, international comparison reveals that students’ performance in measurement in the United States is poor (Clements & Sarama, 2021; Gavin et al., 2013). To be internationally competitive, students must learn “how to measure” and “what and why to measure” concepts related to length, area, and volume (Tan-Sisman & Aksu, 2016, p. 1310).

RESEARCH-BASED RECOMMENDATIONS:

- Provide real-world activities that promote a conceptual understanding of **measurement** concepts and processes (length, weight, angles, perimeter, area, surface area, volume, and time), problem solving, and application (Gavin et al., 2013; NCTM, 2000; Seah & Horen, 2020; Tan-Sisman & Aksu, 2016).
- Build on students’ understanding of linear measurement to support their understanding of **perimeter** and **area** (Tan-Sisman & Aksu, 2016). Emphasize perimeter as a length and draw attention to all four sides to help students understand the formula (P = l + w + l + w) (Van de Walle et al., 2018b). Help students make sense of area by having them cover the surface of 2-D shapes. Encourage students to apply their understanding of multiplication using arrays, grids, or square tiles to find the area of rectangles, before introducing more complex shapes or the formula (A = L x W; A = b x h).
- Capitalize on technology so students can use virtual models, manipulatives, and tools to explore **surface area** and **volume**, which are concepts that have traditionally led to misconceptions (e.g., using the volume formula to determine surface area) (Obara, 2009; Schenke et al., 2020). Provide activities that encourage students to build a structure of solids using unit cubes (virtual or physical) (Battista, 2007) and opportunities to fill, pack, and compare to understand volume as a measurable quantity (Sarama et al., 2011). Before introducing formulas, allow students to develop a deep understanding of these concepts to avoid the ineffective use of strategies (Tan-Sisman & Aksu, 2016).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ understands that many students struggle when first introduced to concepts like **perimeter** and **area**. Therefore, lessons help students understand the mathematics underlying these concepts by integrating real-world contexts, modeling problem solving, and presenting diverse strategies. For instance, in one lesson focused on perimeter as a linear measurement of one-dimensional units (Figure 148), instructional videos remind students what the definition of perimeter is, alongside a strategy that could be used to find the perimeter of a given shape (Figure 149). Lessons focused on measuring the area of a 2-D figure, an important concept that directly relates to multiplication, help students conceptualize what it means to find the total number of same-sized units that cover (or are inside) a figure. This encourages students to make sense of the composition and structure of the shape using square tiles or arrays (Figures 150 and 151) before they engage in more challenging problems, like how shapes with different dimensions can have equivalent areas (Figure 152) or finding the area of more complex figures.

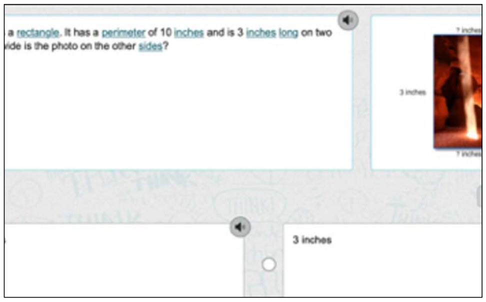


Figure 148. *G.3-Item 4040*

Figure 149. *G.3-Item 4040*

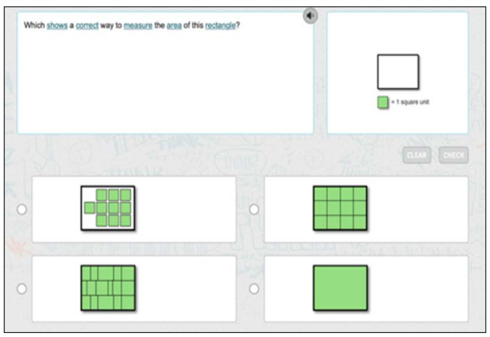


Figure 150. *G.3-Item 3201*

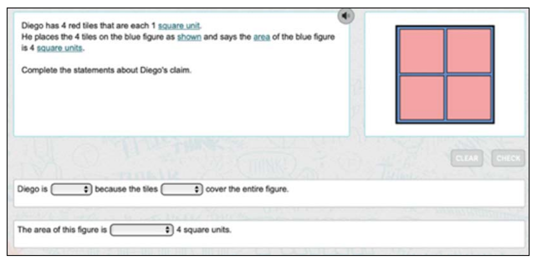


Figure 151. *G.3-Item 4040*

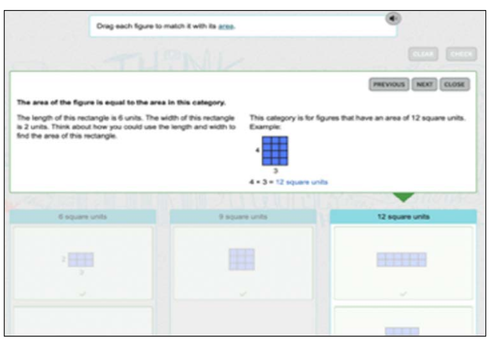


Figure 152. *G.3-Item 94037*

Relatedly, **IM 3+** helps students develop a conceptual understanding of **surface area** and **volume** before introducing formulas. For instance, students learn to determine the surface area of a solid by using nets and finding the area of each surface (Figure 153). Students also learn to apply their understanding of surface area and volume to the real world by exploring relevant and familiar problem contexts (Figure 154). To ensure problems are cognitively demanding, **IM 3+** poses problems like "Drag two figures to the box whose combined volume is greater than 30-unit cubes" (Figure 155), rather than providing the dimensions. As students progress in their learning pathway, they begin to engage in more abstract problem solving. For instance, students translate the volume of a figure into a symbolic expression. Informative feedback is used to reinforce different strategies that could be used to find the volume of this figure (e.g., decomposing the figure into two rectangular prisms) (Figure 156). These lessons help students understand the procedure conceptually and prepare them to use formulas appropriately when determining the volume of figures with whole number and fractional edge lengths (Figure 157).

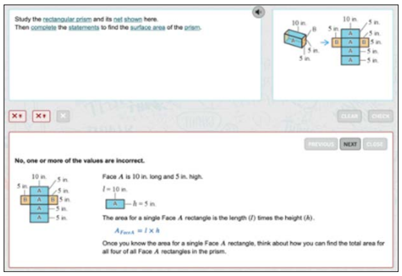


Figure 153. *G.6-Item 34668*

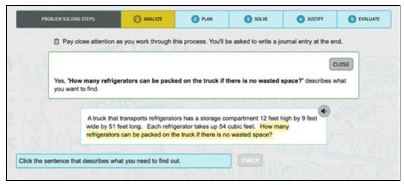


Figure 154. *G.6-Item 70524*

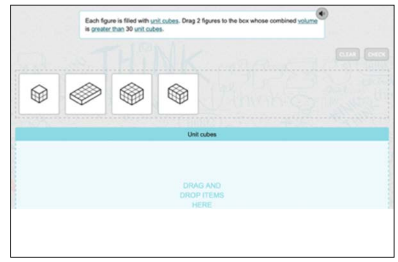


Figure 155. *G.6-Item 94269*

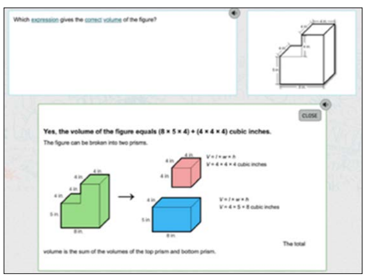


Figure 156. *G.6-Item 4271*

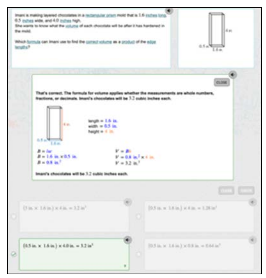


Figure 157. *G.6-Item 34569*

DATA ANALYSIS, PROBABILITY, AND STATISTICS

WHAT THE RESEARCH SAYS:

The digital revolution has made the demand for good data sense and statistically literate citizens even greater (Bargagliotti et al., 2020). **Statistical reasoning** is defined as "the way people reason with statistical ideas and make sense of statistical information" (Garfield & Chance, 2000, p. 101). For students to be prepared to work with data, they need to apply statistical reasoning to familiar and everyday situations (English, 2012; Scheaffer & Jacobbe, 2014). Students develop statistical reasoning skills as they investigate real-world problems and meaningfully interact with data (English, 2013). As students pose questions and **collect, organize, represent,** and **analyze** data, they develop a foundational understanding of statistics (Bush et al., 2014/2015). Data analysis is particularly important because of its role in algebra; it requires students to learn how to collect data, examine patterns and functions, make sense of the relationships, and represent the relationships using multiple representations (e.g., tables, graphs) (Bay-Williams, 2001).

Leaders in mathematics education recommend that **probability** and **statistics** assume a "deeper and wider role" in elementary mathematics curricula (Leavy et al., 2018). The traditional approach to teaching these concepts has often overemphasized mathematics procedures, which are considered ineffective for learning to reason about statistics intuitively (Biehler et al., 2013). Not surprisingly, research has found students have difficulty understanding these concepts (Bryant & Nunes, 2012; Rahmi et al., 2021) and are not learning statistical skills at the level that is needed to make sense of data encountered in everyday life (Glancy et al., 2017). While statistics is about numbers, it is about numbers in context (Scheaffer, 2006). Technology-enhanced instruction can help students learn probability and statistics concepts at a more intuitive level (Leavy & Hourigan, 2015; Makar, 2014; Meletiou-Mavrotheris & Paparistodemou, 2015). Utilizing technology makes statistics visual, interactive, and dynamic; helps emphasize concepts over computation; and offers engaging opportunities to analyze data (Biehler et al., 2013).

RESEARCH-BASED RECOMMENDATIONS:

- Increase STEM activities to expand how students learn and apply **statistical reasoning** (Glancy et al., 2017). Make connections between context-specific applications of data analysis and the concepts that make up statistics, instead of viewing data as simply numbers (Bush et al., 2014/2015). Integrate opportunities for students to respond to high-level questions. Rather than asking "What is the interquartile range?" ask, "Do you think the results of your data collection are representative of students at other schools? Why or why not?"
- Provide opportunities for students to explore the process of **data collection, representation,** and **analysis**. They should investigate real and motivating data sets; organize and display data using different representations (e.g., tables, charts, graphs); study statistical concepts beyond measures of center (e.g., variability, informal inferential reasoning); employ different statistical techniques and tests; and engage in rich discussions that encourage communication of evidence-based conclusions (Biehler et al., 2013).
- Deepen students' analytic skills by incorporating real-world activities that promote a conceptual understanding of **probability** and **statistics**, such as distribution, sampling, measures of central tendency, variability, and predications and inferences (Burrill & Biehler, 2011; Leavy et al., 2018). Capitalize on the use of technology when exposing students to these concepts (Biehler et al., 2013).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ provides opportunities for students across grade levels to expand how they learn and apply **statistical reasoning.** In the following seventh-grade STEM-focused Application Task, students use proportional reasoning to analyze data collected from deer populations (Figure 158). They identify what the data represent, modify missing or corrupt data, organize their data, and make interpretations about factors that would increase or decrease the populations. By incorporating context and encouraging students to apply their thinking, the task helps students make sense of statistics concepts beyond simply viewing data as a number. This task also helps students make connections across different mathematics concepts. Students explore important algebra concepts, such as patterns and functions, making sense of relationships and communicating these relationships using different representations (e.g., tables, graphs).

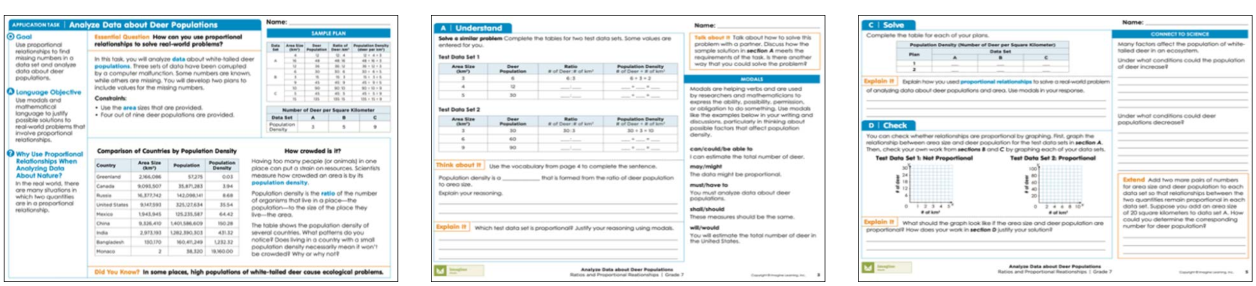


Figure 158. *G.7-Analyze Data about Deer Populations.*

In **IM 3+,** students explore a range of concepts, including measures of central tendency, distribution, variability, sampling, probability, and inferences. They interact with different representations to make sense of these concepts. For instance, fifth-grade students analyze data in a scatter plot to determine the meaning of a point and use the trend in the data to make predictions (Figure 159). They also explore statistical variability by answering questions about the center, spread, and overall shape of the data (e.g., identifying quartiles in a set of data) (Figure 160). Lessons purposely incorporate relevant data so that learning is meaningful to students. Lessons encourage students to **collect, represent,** and **analyze** data using different forms of representation. For instance, students analyze a box plot and make inferences about the effectiveness of a video game. These lessons help students understand data in context and the relevance of these concepts to their everyday lives (Figure 161).

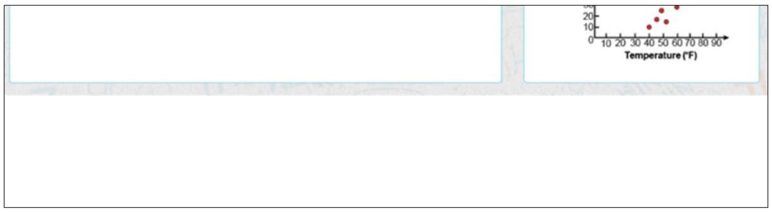


Figure 159. *G.5-Item 1432*

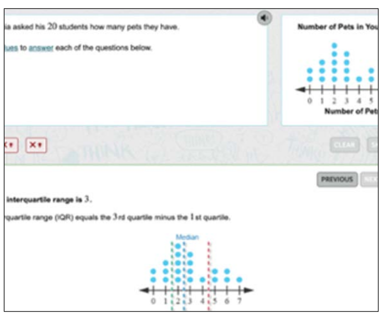


Figure 160. *G.6-Item 4777*

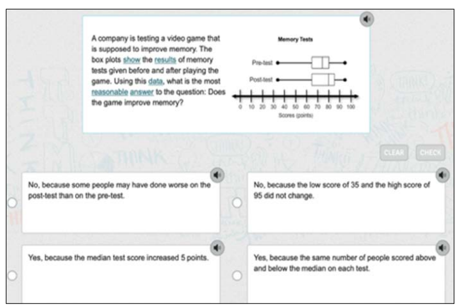


Figure 161. *G.7-Item 3780*

IM 3+ capitalizes on the use of technology to teach **probability** and **statistics** concepts. Lessons are intentionally designed to help students develop into statistically literate citizens who can work with data in everyday situations. Students explore various statistical methods, connect data to chance, engage in inferential reasoning, and draw conclusions. For instance, in a seventh-grade lesson on sampling, students determine the most appropriate way to collect survey data on a classroom field trip. In doing so, they learn how to analyze relevant data from a sample and learn to draw inferences about populations (Figures 162 and 163). This lesson also emphasizes key concepts like random sampling to discuss the importance of drawing unbiased conclusions and generalizing results to a larger population. **IM 3+** seamlessly integrates algebra content with important statistics concepts (e.g., normal distribution, bell curve, standard deviation, mean). For example, in an Algebra I lesson, students analyze data collected from a class survey and apply their understanding of measures of center, standard deviation, and distribution (Figure 164).

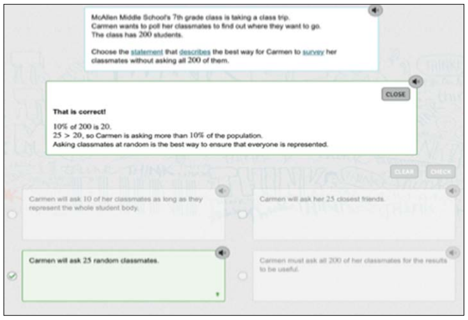


Figure 162. *G.7-Item 34713*

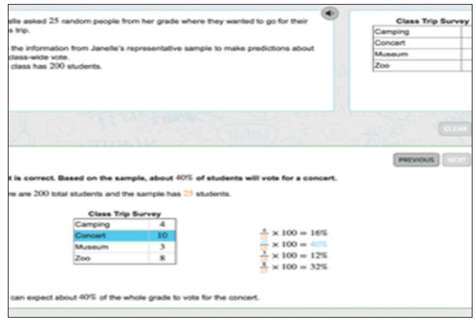


Figure 163. *G.7-Item 34713*

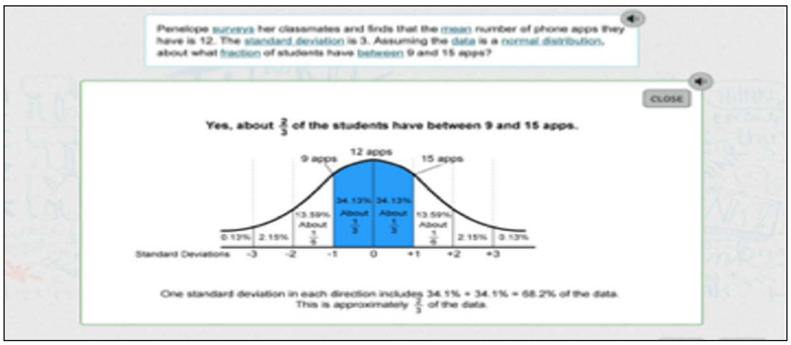


Figure 164. *Algebra-Item 50306*

Principle 2: Integrate research-based mathematics teaching practices that encourage problem solving, reasoning, and real-world application.

WHAT THE RESEARCH SAYS:

To foster systemic improvements in mathematics education, the adoption of a set of effective instructional practices helped address the challenges associated with teaching and learning mathematics, such as accessibility and equity (NCTM, 2014a). These practices include a coherent curriculum; the integration of cognitively demanding tasks; the use of multiple representations; opportunities for meaningful discourse; and building procedural fluency from conceptual understanding.

Coherence

The National Mathematics Advisory Panel (2008) defines a **coherent** curriculum as “effective, logical progressions from earlier, less sophisticated topics into later, more sophisticated ones” (p. xvii). A coherent, logically sequenced curriculum that maintains the consistency of content and lessons, within and across grade levels, is important for nurturing a conceptual understanding of mathematics (Cai et al., 2014; NCTM, 2014b). Rather than focusing on a long list of concepts or skills within a lesson, students benefit from instruction that focuses on one central idea, which they can then connect to other concepts learned (Merritt et al., 2010).

Cognitively Demanding Tasks

Studies show that students who more readily engage in **cognitively demanding tasks** demonstrate a deeper conceptual understanding of mathematics (Cai et al., 2011) and greater concept mastery than students who predominantly solve procedural tasks (Boaler & Staples, 2008). Cognitively demanding tasks encourage students to engage in complex thinking, while procedural tasks emphasize facts, memorization, and procedures without connections to underlying concepts (Jackson et al., 2013; Stein et al., 2009). Tasks high in cognitive demand provide multiple entry points, promote diverse strategies and solutions, encourage reasoning about relationships, foster connections across representations, and promote student explanations.

Cognitively demanding tasks also foster **productive struggle**, or the struggle to make sense of unfamiliar concepts and procedures that are not immediately apparent (Hiebert & Grouws, 2007). Students who engage in productive struggle delve "more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions" (NCTM, 2014b, p. 48). While these tasks may be challenging, they fall within a student's ability to solve without the direct help of a teacher (Smith et al., 2018). Experts agree that productive struggle is a critical part of the learning process because it encourages persistence in problem solving, leads to a stronger conceptual understanding, fosters agency, and improves metacognitive strategies (Kapur, 2014; Sinha & Kapur, 2021). This is important because students who have access to tasks that encourage productive struggle have opportunities to engage in deep mathematical thinking, high-level reasoning, and problem solving (Huinker & Bill, 2017; Lynch et al., 2018).

Discourse

Studies show positive associations between **mathematical discourse** that emphasizes reasoning and problem solving and student learning outcomes (Michaels et al., 2008). According to Smith and Stein (2018), mathematical discourse provides benefits for students across grade levels, including those with learning disabilities and struggling in mathematics. Discourse encourages students to explain and justify ideas, clarify understanding, connect prior knowledge to new concepts, address misconceptions, and build a shared understanding of mathematics concepts. These opportunities are important because they teach students how to communicate clearly, while strengthening their conceptual understanding of key mathematics concepts. They also provide a platform to discuss mistakes and misconceptions, which—when positioned as opportunities to learn and improve—can lead to a healthy growth mindset (Boaler, 2016). For a deeper discussion on the importance of mathematical discourse, please see Principle 3 (p. 60).

Multiple Representations

Research recommends the use of **multiple representations** (contextual, visual, verbal, physical, and symbolic) to support students' understanding of concepts and procedures (NCTM, 2014b). Studies show positive effects of students' use of multiple representations and their conceptual understanding (Ainsworth, 2006; Rau et al., 2009). Yet, many students struggle to negotiate the different forms and functions of representations (Heinz et al., 2009). Helping students make connections across representations (e.g., diagrams, tables, models, equations, real-world situations) fosters a deeper understanding of the underlying mathematics concepts (Dreher et al., 2016; Duval, 2006), develops flexibility in use when solving problems, and teaches them to organize their ideas to communicate effectively (Huinker & Bill, 2017).

Relatedly, mathematical literacy is critical for our society, and students need opportunities to apply mathematics to their everyday lives (Wijaya et al., 2015). Real-world contexts, or contextual representations, are a powerful tool for promoting problem solving (NCTM, 2014b). Real-world contexts are often presented as word problems, which provide a meaningful basis for students to transform the context of a situation into a mathematical form. Grounding mathematics in contexts that are relevant to students (Ladson-Billings, 2009) enriches their understanding (Van de Walle et al., 2018b), honors their lives outside of the classroom, affirms their cultural experiences (Ukpododu, 2011), and promotes agency (Schoenfeld, 2014).

Conceptual Understanding and Procedural Fluency

There has been a clear movement in research and practice toward a more balanced approach to developing students' conceptual and procedural knowledge (Crooks & Alibali, 2014). The concrete-representational-abstract framework (Bruner & Kenney, 1965) is a well-documented approach for supporting students' **conceptual understanding** and helping them develop deep understanding of how mathematics can apply to the real world (NCTM, 2014b). In this approach, students begin by solving problems using concrete objects (e.g., base-10 manipulatives), then using representations like drawings or pictures (e.g., number line), and finally solving problems abstractly (e.g., symbols and numbers). Support is gradually faded as students begin to master the concept (Agrawal & Morin, 2016; Bouck et al., 2018).

A strong conceptual understanding of procedures lays the foundation for **procedural fluency** (Milton et al., 2019; Rittle-Johnson et al., 2015). Procedural fluency involves "knowing how a number can be composed and decomposed and using that information to be flexible and efficient with solving problems" (Parish, 2014, p. 159). Learning to reason flexibly, accurately, and efficiently is important for students' work with whole numbers, fractions, geometry, measurement, and algebra (Huinker & Bill, 2017; National Mathematics Advisory Panel, 2008). However, difficulties arise when procedures are taught prematurely. This can lead to confusion, misconceptions, and a misuse of strategies (Rittle-Johnson et al., 2001). Research urges against instruction that emphasizes memorized facts, speed, and timed tests because these can be damaging and cause mathematics anxiety (Boaler, 2015). Boaler (2014) reminds us, "Learning is a process that takes time, and it cannot be accelerated by methods that encourage speed at the expense of understanding" (p. 473).

RESEARCH-BASED RECOMMENDATIONS:

- Develop a **coherent** curriculum that is based on students' learning progressions. The curriculum should be logically sequenced in order to help students make meaningful connections across mathematics concepts, units, and grade levels (Huinker & Bill, 2017).
- Provide students with tasks that are **cognitively demanding** and require cognitive effort. These tasks should promote multiple solution pathways, connections across concepts, multiple representations, reasoning, problem solving, explanations, and justifications (Stein et al., 2009).
- Support **productive struggle** by providing task scaffolding (multiple forms of representation, real-world contexts, graphic organizers, defined key terms) and teacher encouragement (e.g., praise) to increase student learning and motivation, including English language learners (ELLs) and students with special needs (Townsend et al., 2018).
- Promote **mathematical discourse** in an online learning environment by encouraging students to explain and justify their thinking through verbal, visual, and written forms of communication. Actively engage students in the learning process by incorporating purposeful questions ("What strategy might you use to solve this problem?") or concrete support for students who are struggling ("Could you draw a picture to help you solve this problem?") (Harbour & Denham, 2021).
- Build students' understanding of concepts and procedures by using **multiple representations**

(contextual, visual, verbal, physical, and symbolic) to help them make explicit connections to the concept. This encourages flexibility in students' selection and their purposeful use of representations to solve a given mathematical situation (Dreher et al., 2016; Marshall et al., 2010). Integrate problems that contain culturally responsive, **real-world contexts** to make learning relevant for students (NCTM, 2014b).

- Promote **conceptual understanding** by incorporating the concrete-representational-abstract framework into the design of lessons (Agrawal & Morin, 2016; Bouck et al., 2018; Flores, 2010). Gradually fade support as students learn to master the concept or skill.
- Help students develop **procedural fluency** by building their conceptual understanding of mathematics concepts, relationships, and operations over time. Activate students' prior knowledge, incorporate real-world contexts, encourage multiple representations and strategies, and promote discourse to help students make connections between concepts. These practices inform students' flexible use of procedures and ability to determine what strategy is most appropriate when solving a mathematical problem (Bay-Williams, 2020; Hendrickson et al., 2018).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ is more than a collection of lessons or topics; it is a coherent supplemental program organized around student learning progressions. It contains logically sequenced lessons that help students develop a strong understanding of content before moving on to more advanced concepts and skills. For instance, **IM 3+** recognizes topics related to place value can be difficult, because they require students to integrate their conceptual knowledge of the base-10 system with procedural knowledge of how place value concepts are represented, written, and spoken. Therefore, **IM 3+** reinforces these concepts across different units and grade levels (e.g., whole-number place value concepts, estimation, rounding, number magnitude, and operations). Each lesson in a student's pathway carefully connects to and builds on what they learned in earlier grades. In the early elementary grades, students use models to represent a given number (e.g., virtual base-10 manipulatives) and learn to attend to the value of each digit in a number (e.g., 100s, 10s, and ones). The use of precise written and spoken language makes place value concepts explicit (there are four 10s and three ones in the number 43). Students first explore groups of 10 and 100, then groups of 1,000 and beyond. They learn to reason about the relationships between numbers (e.g., a digit in the one's place represents 10 times as much as it represents in the place to the right). As students progress to the upper elementary and middle grades, they draw on their knowledge of whole-number place value as they explore decimals, measurement, and money. These lessons focus on concepts like understanding the 10-to-1 multiplicative relationship between values of two adjacent positions; estimating, comparing, adding, and subtracting decimals; place value positions within the metric system; and the role of the decimal point in the U.S. monetary system.

IM 3+ integrates **cognitively demanding** tasks that promote a conceptual understanding of mathematics. These tasks encourage students to reason about concepts and relationships, use different strategies, make connections across multiple representations, and explain and justify their thinking. For example, in a seventh-grade Application Task, "Plan a Boat Trip on the Nile River," students compute speeds of wind-powered boats as unit rates, which they use to plan a trip on the Nile River (Figure 165). This task requires a considerable amount of cognitive effort and engages students in the process of doing mathematics. Students build background knowledge on the topic, select and solve the problems using their own strategies, use and make connections across multiple representations (e.g., tables and charts to organize their thinking, equivalent expressions, written responses), and reason about their solution through verbal and written explanations and justifications. The task encourages students to think about rate conceptually, by asking students to explain why a procedure works (*"dividing the distance by the time to find the unit rate"*) (Figure 166) and how they used their knowledge of proportional relationships to solve a real-world problem (Figure 167). The extension (*"Add an additional boat and determine how the total time of the trip will be impacted"*) creates an additional challenge for students to maintain engagement among all learners.

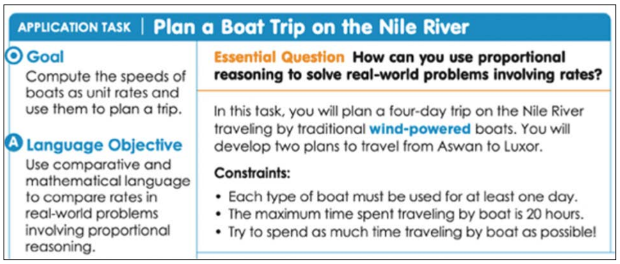


Figure 165. *G.7-Plan a Boat Trip on the Nile River*

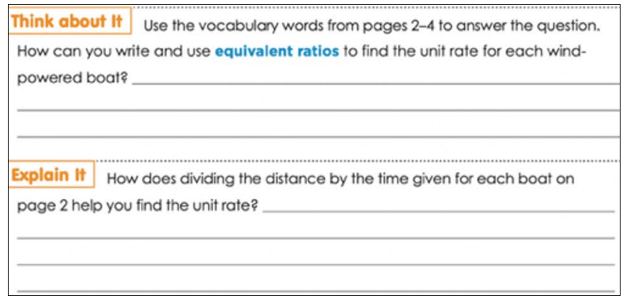


Figure 166. *G.7-Plan a Boat Trip on the Nile River*

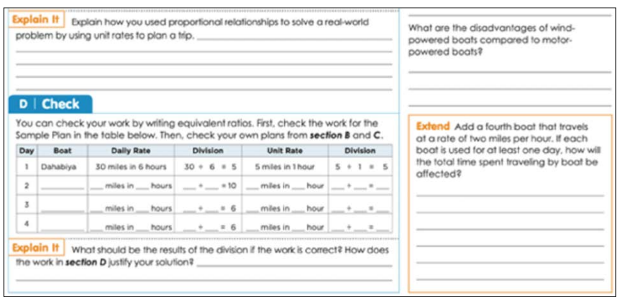


Figure 167. *G.7-Plan a Boat Trip on the Nile River*

IM 3+ fosters **productive struggle** within a safe and supportive learning environment. Cognitively demanding tasks encourage students to grapple with unfamiliar concepts and procedures, which may not be immediately apparent. When students have difficulty solving the problem without instructional support, **IM 3+** lessons provide multiple levels of scaffolding. First, students receive immediate, informative feedback based on their response. Then, students can also access two Math Helps, which provide hints and information on how to solve the problem. For instance, one Math Help might provide a representation to help students visualize a concept (e.g., Figure 168). If students use both Math Helps and still need assistance, they can interact with **Imagine Math's** Live Teachers. Live Teachers are certified mathematics teachers who deliver individualized support in English or Spanish. While students can actively seek out the support, the program has a built-in system that proactively intervenes so that students are offered the right amount of help at the right time. Students will receive a popup notification asking if they would like to interact with a Live Teacher if they are working on a lesson for the second time, scored less than 60% on a Pre-Quiz, or incorrectly answered a problem in Guided Learning. During these interactions, the teacher and student can utilize a two-way interactive whiteboard (Figure 169), which simulates a classroom environment and allows for more intensive instruction.

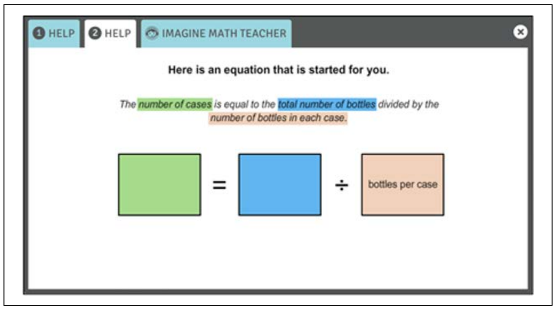


Figure 168. *Math help*

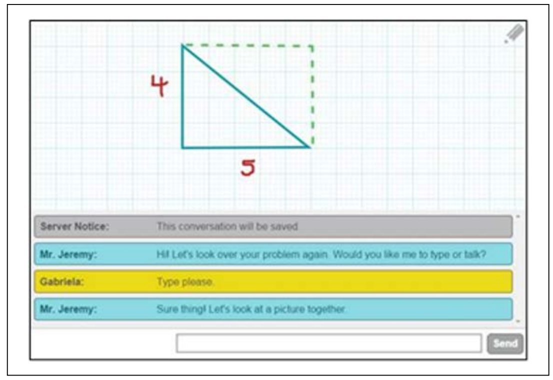


Figure 169. *Interactive whiteboard*

IM 3+ promotes **mathematical discourse** by prompting students to explain and justify their ideas, connect prior knowledge to new concepts, clarify understanding, and communicate their thinking effectively. A unique feature of the **IM 3+** learning environment is that students are empowered to talk with a Live Teacher during each lesson. An excerpt of a conversation between a teacher and student is provided below. This example illustrates how the teacher elicits the student's thinking to promote rich mathematical discourse. The teacher asks purposeful questions to help the student make connections to their prior fraction knowledge (Figure 170). The teacher also encourages the student to represent their thinking using a visual model on the interactive whiteboard to address their misconception. The teacher continues to model clear and concise language by restating what the student has said and emphasizing key concepts when communicating to direct students' attention to the mathematics goal.

The diagram shows three identical 1-gallon buckets with different amounts of paint. About how much paint is there in all? Choose the answer that makes the most sense.
Teacher: I love how you thought about that. You're already thinking of putting all three containers together. Let's think about what that fraction 5/6 would mean. If you wanted to put that on a number line, please show me where 5/6 would be located.
[Student creates number line]
Teacher: Very nicely done. What do you notice about how that amount compares to one whole?
Student: With 5/6 there is 1/6 left until the whole.
Teacher: Awesome. So you're seeing that your estimate is that you would have less than one whole container filled. Please share some other fractions that you know are just a little less than one whole.
Student: 2/6
Teacher: Those are great examples of fractions that are all less than one whole. Let me ask this: Is 2/6 closer to zero or one whole?
Student: Zero.
Teacher: How did you think about that?

Figure 170. *G.5-Item 4324*

IM 3+ encourages students to use **multiple representations** to develop flexibility in their selection and use of a representation and develop a conceptual understanding of the concept. For instance, a lesson focused on understanding fractions as division exemplifies how **IM 3+** integrates multiple representations to support students' conceptual understanding of mathematics. In Figure 171, students are presented with a real-world problem (contextual representation) where they are asked to solve an equal sharing problem (written description): *"Matt has 12 balloons that he wants to give to 4 friends. He wants each friend to have the same* *number of balloons."* Students select from written statements (*"Each friend gets 12/4 balloons"*) and symbolic representations (*"12 ÷ 4"*) when determining the answer. Immediate feedback also reaffirms students' answer choices using written and verbal explanations, as well as visual models (Figure 172).

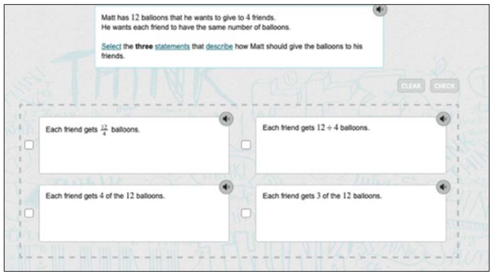


Figure 171. *G.5-Item 33645*

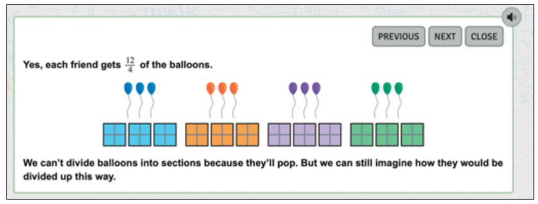


Figure 172. *G.5-Item 33645*

IM 3+ knows that **procedural fluency** is built from **conceptual understanding**. For instance, becoming fluent with multiplication facts is an important part of students' development. **IM 3+** lessons incorporate problems that focus on supporting students' understanding of the underlying concepts behind the procedures. Students use concrete objects to explore the meaning of equal groups (Figure 173), make connections between multiplication and repeated addition, and represent problems using arrays (Figure 174) and area models (Figure 175). Students also have opportunities to apply their understanding of the properties of operations, another important part of building procedural fluency. Students explore the commutative property as they solve a problem involving an array, and they learn about the distributive property over addition when they solve a problem using an area model (Figure 176). Students' experiences with these strategies strengthen their fluency and prepare them to solve multi-digit problems (Figure 177). This is especially important because students who apply their understanding of the area model can solve multi-digit problems more fluently and make sense of the partial products method more conceptually (Figure 178).

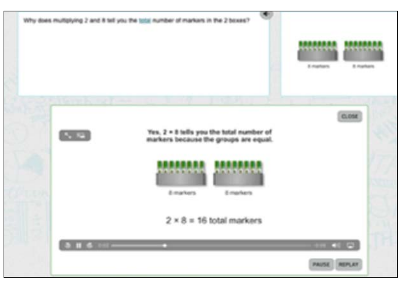


Figure 173. *G.3-Item 7970*

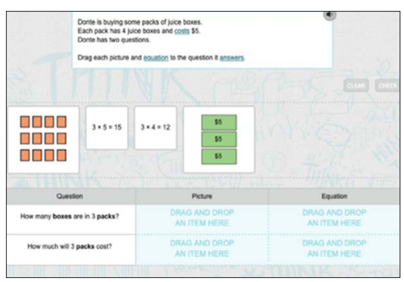


Figure 174. *G.4-Item 30329*

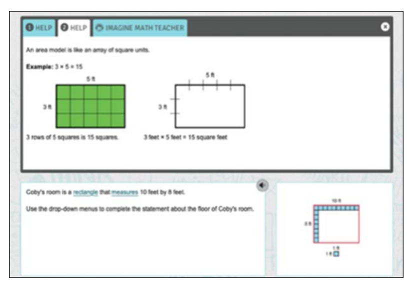


Figure 175. *G.4-Item 30330*

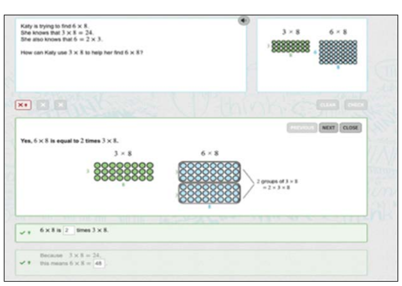


Figure 176. *G.3-Item 41291*

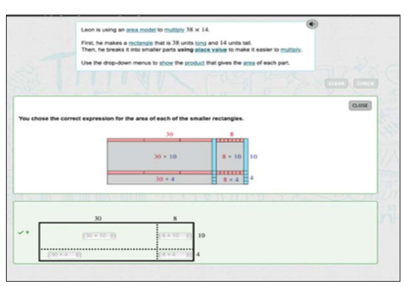


Figure 177. *G.4-Item 41071*

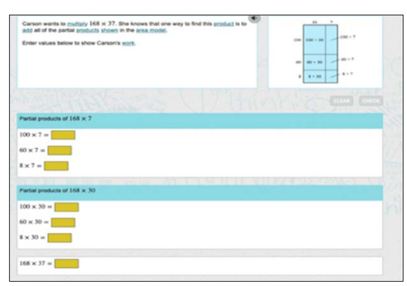


Figure 178. *G.5-Item 33071*

Building a deep conceptual understanding of mathematics takes time. That is why **IM 3+** incorporates the concrete-representational-abstract framework. Using multiplication as an example again, students are first introduced to concrete objects (e.g., virtual markers, counters, base-10 blocks) to represent and reinforce the concept of equal groups (Figure 179). As students progress in their learning pathway, they explore and use visual representations, such as arrays and area models (Figure 180). Finally, once students develop a more abstract understanding of multiplication, they are exposed to standard algorithms (Figure 181).

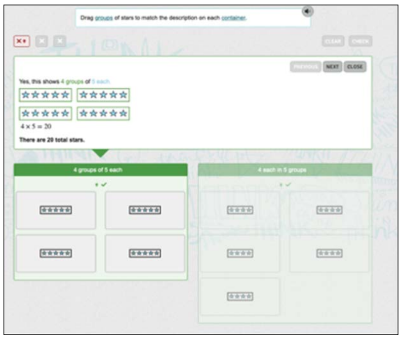


Figure 179. *G.3-Item 94598*

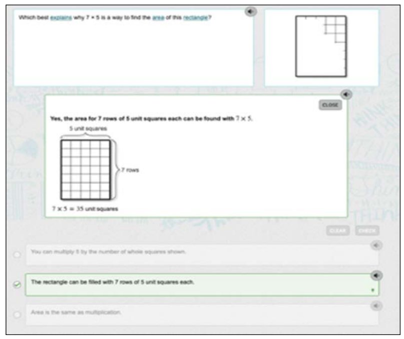


Figure 180. *G.3-Item 3219*

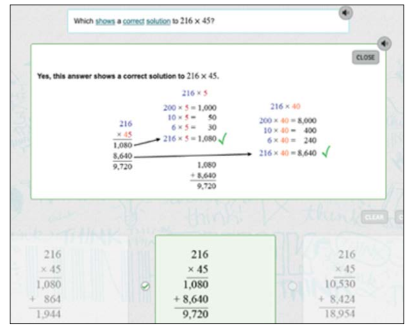


Figure 181. *G.5-Item 33072*

Principle 3: Promote mathematical discourse to help students develop effective communication skills and a deep understanding of mathematics.

WHAT THE RESEARCH SAYS:

Researchers agree that discourse, language, and vocabulary play a critical role in learning mathematics (Groth, 2013; Sammons, 2018; Vukovic & Lesaux, 2013). NCTM (2014b) describes **mathematical discourse** as "the purposeful exchange of ideas through classroom discussion, as well as other forms of verbal, visual, and written communication" (p. 24) and a "primary mechanism for developing conceptual understanding" (p. 30). Engaging in mathematical discourse is important because it empowers students to use multimodal forms of communication (e.g., verbal, written, pictures) to share their ideas, justify their thinking, make connections, critique the reasoning of others, and refine their thought processes. It also illuminates what students understand about a particular concept, as well as potential misconceptions. Without access to discourse, opportunities to learn mathematics are significantly reduced (Banse et al., 2016).

For many students, learning mathematics is similar to learning a new language. Learning to communicate effectively "through the language of mathematics requires mathematical understanding; a robust vocabulary knowledge base; flexibility; fluency and proficiency with numbers, symbols, words, and diagrams; and comprehension skills" (Riccomini et al., 2015, p. 237). Research suggests teaching students to use and apply the language of mathematics in oral, written, and representational forms leads to improvements in reasoning, conceptual understanding, and discourse skills (Moschkovich, 2013; Riccomini et al., 2015).

Mathematics **vocabulary** is also an important component of instruction (Lin et al., 2021; Seethaler et al., 2011) and can predict students' performance in mathematics (van der Walt, 2009). However, everyday and academic vocabulary acquisition can be challenging for students, particularly ELL students, students with disabilities, and students struggling in mathematics (Bay-Williams & Livers, 2009). For instance, mathematical concepts are often expressed in multiple ways (e.g., sum, add), vocabulary words are specific to mathematical contexts (e.g., parallelogram, composite number), and translating context-specific words from one language to another can be difficult (e.g., *mesa*, table) (Riccomini et al., 2015). Experts contend that mathematics vocabulary should be taught explicitly (Sammons, 2018). With consistent opportunities to strengthen **mathematical discourse**, **language development**, and **vocabulary**, students' communication deepens, broadens, and becomes increasingly complex.

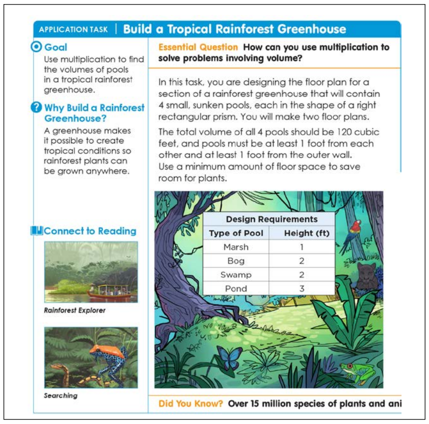
RESEARCH-BASED RECOMMENDATIONS:

- Promote **mathematical discourse** by engaging students in cognitively demanding tasks that contain several solution pathways and encourage multiple forms of communication (e.g., verbal, visual, and written). These tasks should elicit student thinking, ask high-level questions, encourage students to listen to and critique the arguments of others, and use precise vocabulary (NCTM, 2000).
- Incorporate talk moves, such as **probing questions** (e.g., how and why) that ask students to explain, elaborate, or clarify their understanding of mathematics concepts, and **specific questions** (e.g., explicit and direct) to draw attention to critical mathematics content and scaffold learning (Banse et al., 2016).
- Foster students' **language development** by promoting their use of words, symbols, and models to represent their mathematical thinking, make sense of their ideas, and clarify their understanding (Huinker & Bill, 2017). Opportunities like journal writing help students learn to express their understanding of vocabulary through written text. Graphic organizers help students communicate using multiple representations (e.g., equations, models, examples and non-examples).
- Support **vocabulary** development by activating students' prior knowledge and connecting it with new vocabulary words (Riccomini et al., 2015). Clearly present and model word meanings using relevant contexts and provide repeated and meaningful opportunities to apply the meaning of new words (Bay-Williams & Liver, 2009). Sentence starters, graphic organizers, word walls, and personal glossaries help scaffold students' understanding and use of new vocabulary.

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ fosters **mathematical discourse** by engaging students in tasks that promote the use of multiple representations, ask high-level questions to support students' acquisition and use of precise mathematics language, and prompt students to explain their strategies and solutions. In a fifth-grade Application Task, "Build a Tropical Rainforest Greenhouse", students are asked how they could use multiplication to determine the volume of four pools in a rainforest greenhouse (Figures 182). The open-endedness of this Application Task allows students to employ diverse strategies to determine the volume of the pools they design. Students make connections across different concepts (measurement, geometry, whole-number operations) and express their thinking using multiple modalities (e.g., completing a table with the appropriate measurements and providing a written explanation of the strategy they used to solve the problem). The Application Task provides a glossary of academic (e.g., greenhouse) and mathematics vocabulary (e.g., cubic unit) words. Students are encouraged to use these words in response to written prompts such as, "*How did you find the length and width for each pool?*" There are also opportunities to engage in discourse in the "*Talk About*

It" sections. Students are invited to make their mathematical thinking visible by collaborating with their peers to design different floor plans for a greenhouse, comparing their designs, identifying patterns in their measurements, and redesigning a floor plan with different dimensions as an extension.



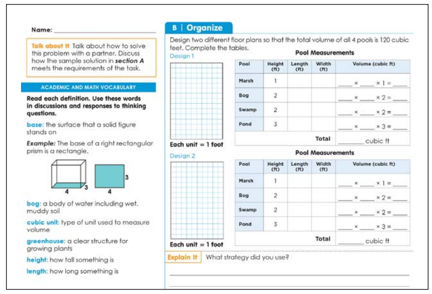


Figure 182. *G.5-Build a Tropical Rainforest Greenhouse*

During a lesson, students can access one of **Imagine Math's** Live Teachers for intensive, one-on-one support. During this time, the teacher and student can communicate using their voice, the chat box, and a two-way interactive whiteboard. The teacher can view the students' work to pinpoint areas of difficulty and pose purposeful **questions** to encourage students to explain their thought process. For instance, rather than asking a question that elicits a single-word answer, "I ate 10% of my pizza and you ate 25% of your pizza. If our pizzas were the same size, who ate more?" teachers ask questions like, "If I ate 10% of a pizza and you ate 25% of a pizza, what is one way I could have eaten more pizza than you? Draw a picture and write an equation to justify your thinking." The transcript below (Figure 183) illustrates a conversation between a real student and Live Teacher. Notice how the teacher elicits the student's thinking using probing questions to encourage predictions, explanations, and justifications, rather than proposing their use of a specific procedure to solve the problem. These teachers are trained to facilitate productive mathematical discourse, which includes the use of appropriate talk moves and rich questioning techniques to help guide students toward a conceptual understanding of the content.

Teacher: What do you wonder about what I wrote?
Student: Why is there 20 ounces of chicken? Did you can it? How does this relate to the question?
Teacher: Great questions. This is an example that will help you understand how the numbers are related. How many servings do you predict I can make?
Student: It depends...how many ounces are in a serving?
Teacher: You are asking great questions. Suppose I want 4 ounces in each serving.
Student: Okay, you can make 5 servings.
Teacher: Terrific! How did you figure that out?
Student: 4 x 5 equals 20 and you had 4 ounces of chicken in each serving.
Teacher: How else can we figure out how many servings we can make from 20 ounces of chicken?
Student: 20 divided by 5 equals 4.

Figure 183. *Conversation with Live Teacher*

IM 3+ promotes **language development** by encouraging the use of multimodal forms of communication (e.g., written and oral language, symbols, models, virtual manipulatives). This helps students build meaning, clarify understanding, compare strategies, and form connections across concepts and representations. In Figure 184, students solve the problem, *"Bottled water comes in cases of 24 bottles. You need 150 bottles of water for a school event. How many cases of water do you need to buy?"* **IM 3+** recommends several different strategies (*"Draw a picture, diagram or model,"* or *"Look for a pattern"*), which illustrate the number of ways a student can solve the problem. Six digital manipulatives are also available for students to use (bar models, number lines, fraction pieces, fraction shapes, area models, base-10 manipulatives). Write or Talk prompts are embedded in the lesson to encourage students to reflect on their problem solving. This prompt asks students to explain their strategy and the reasonableness of their answer. Sentence frames are provided to scaffold students' responses (*"An important part of the solution to the problem is…"* or *"This strategy was effective for the problem because…"*). Linking words and phrases (*"To give an example" or "In contrast"*) are also included to enhance a student's ability to communicate clearly and coherently.

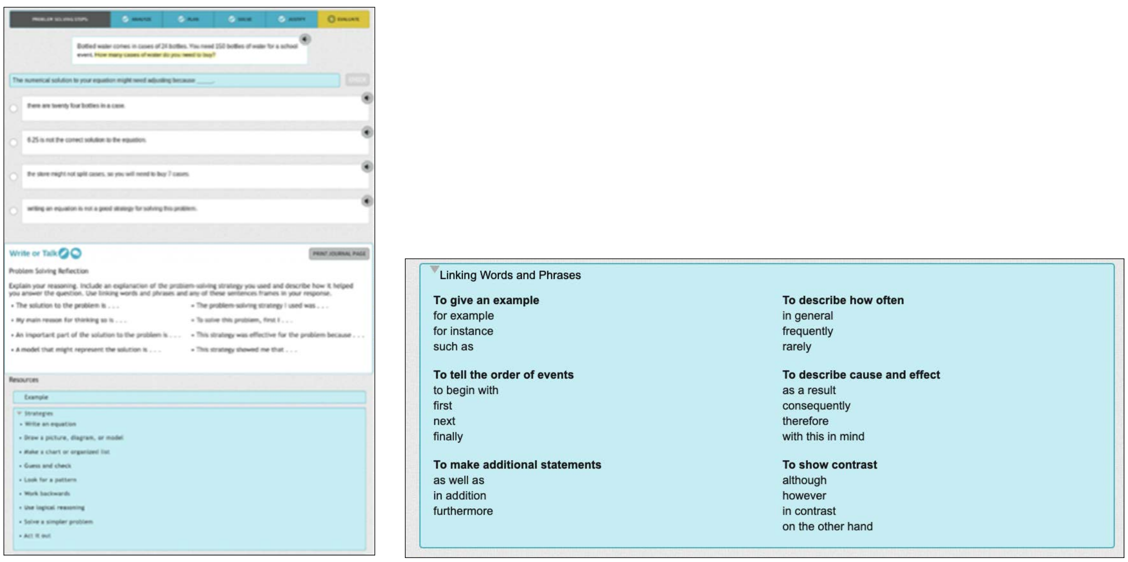


Figure 184. *Figure 184. G.5-Item 70527*

IM 3+ provides Journaling Pages, which are available to students to print and use during any lesson (Figure 185). These resources are available in English and Spanish. They provide space for students to organize their mathematical thinking, take notes on important concepts, write their own definitions of key vocabulary words, solve problems using various strategies, ask questions, and reflect on their learning. In one section of the Journaling Pages, students are encouraged to reflect on the lesson and *"Write at least one important math vocabulary word or phrase that was used in this lesson. For each word or phrase, write the definition in your own words and draw a visual representation."* Students can record their thoughts and responses in their native language, along with a visual representation, to promote stronger language development.

IM 3+ recognizes the importance of vocabulary development. The program supports students' acquisition and use of **vocabulary** by helping them connect prior knowledge to new vocabulary words, providing real-world contexts, modeling word meanings, and providing repeated exposure to new words. Students have access to an interactive glossary of more than 500 essential academic and mathematics vocabulary words, which are available in English, Spanish, Tagalog, Haitian-Creole, Arabic, and Vietnamese. They have the option to interact with the interface and opt to have the words read out loud to them in their chosen language. This glossary strengthens students' vocabulary knowledge by ensuring the definitions, models, and problems are defined accurately and consistently. The glossary also provides visual supports and examples to help students encode meaning. For instance, the term "equivalent fractions" provides a written definition, an area model, symbol notation, and a number line in the glossary (Figure 186). The use of multiple representations helps students develop a conceptual understanding of the concept and learn to translate across different representations.

Figure 185. *Journaling pages*

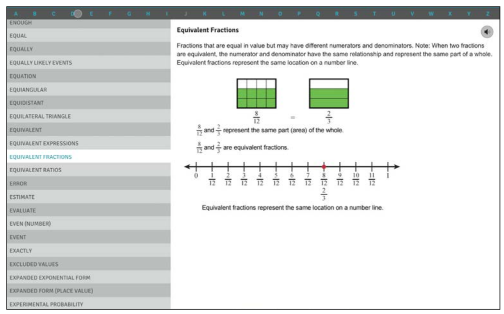


Figure 186. *Interactive glossary*

Principle 4: Ensure all learners receive equitable and accessible mathematics instruction.

WHAT THE RESEARCH SAYS:

Equitable mathematics instruction is an assets-based approach to teaching that recognizes culturally grounded experiences as a foundation to build knowledge (Celedon-Pattichis et al., 2018; Gay, 2000; Kieran & Anderson, 2019). Equitable instruction champions high expectations and provides optimal access to fair opportunities to learn (Gutierrez, 2012). However, many students lack access to high-quality instruction and resources (e.g., teachers, curriculum, technology) (Darling-Hammond, 2001; Flores, 2007). As a result, these inequities fuel disengagement, poor performance, and dropout rates (DeCuir-Gunby et al., 2010; Steele, 2010), and limit possibilities to take advanced coursework, attend college, or pursue careers (Boaler, 2016). Interrupting these inequities helps students reach their full academic potential (National Equity Project, n.d.) and transforms their beliefs about who can achieve in mathematics.

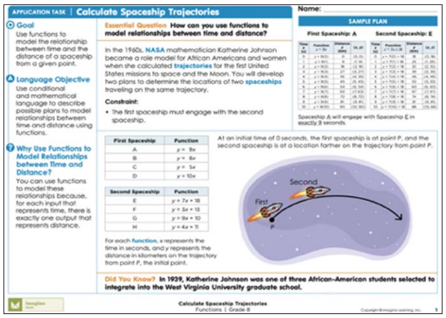
When mathematics is **accessible** to a wide range of students—meaning students can productively and successfully engage with the content—learning outcomes transcend (NCTM, 2014a). To facilitate this, the **Universal Design for Learning** framework is an evidence-based framework that offers instructional guidelines organized around three fundamental principles (**multiple representations, action and expression,** and **engagement**) (CAST, 2018). Integrating UDL principles into instruction ensures students receive scaffolded, adaptive instruction and targeted assessments (Kieran & Anderson, 2019) that fall within their zone of proximal development (Vygotsky, 1978). The term **scaffolding** is often used to describe the types of instructional supports provided to students (e.g., question prompts, feedback, models, graphic organizers). Scaffolding helps students master a concept that they were initially unable to grasp independently (Molenaar & Roda, 2011; West et al., 2019). Research has found scaffolding improves student learning outcomes (Belland et al., 2017; Gersten et al., 2009; Hudson et al., 2006; Smith et al., 2016) and UDL has proven success in reducing barriers and maintaining high achievement expectations for all students (Cook & Rao, 2018).

RESEARCH-BASED RECOMMENDATIONS:

- Promote **equity** in mathematics by taking an asset-based approach to teaching. Connect learning with students' interests and lived experiences, position students as capable learners, maintain high expectations, encourage multiple forms of discourse and language, and foster strong mathematical identities (Bartell et al., 2017; Celedon-Pattichis et al., 2018).
- Ground mathematics teaching in **UDL** principles to support all students regardless of race, gender, socioeconomic status, language proficiency, learning disability, and other social or cultural factors (CAST, 2018; Kieran & Anderson, 2019):
    - Create learning opportunities that are relevant and meaningful.
    - Integrate **multiple forms of representation** to reduce barriers to print, and ensure information is equally perceptible to all students.
    - Incorporate new vocabulary and frequent opportunities to hear and use vocabulary.
    - Encourage the use of diverse tools and multimedia technologies to **express and communicate** understanding of critical ideas.
    - Provide corrective feedback that is clearly and explicitly connected to high standards. Feedback should capitalize on mistakes as opportunities to learn.
    - Encourage persistence, **engagement**, and motivation.
- Make learning **accessible** by providing **adaptive scaffolding** that encourages mastery of content (Gottlieb, 2016; Lei et al., 2020). For instance, students with limited English proficiency benefit from visual scaffolds (e.g., models, graphics) that help them make connections between English words/phrases, the visual image, and the target mathematics concept. Linguistic scaffolds (e.g., bilingual glossaries) also provide support in a language that is comprehensive to the student.

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ promotes **equity** in mathematics by taking an asset-based approach to teaching. **IM 3+** makes a concerted effort to provide learning opportunities that are relevant to students' lives and topics that are worth learning about. The program promotes positive mathematical identities by positioning all students as competent and capable learners. By maintaining high expectations and standards, students have access to rigorous tasks that focus on building conceptual understanding. In this eighth-grade Application Task, "Calculate Spaceship Trajectories" (Figure 187), students engage in functional thinking while investigating intriguing, real-world science content. Students learn how Katherine Johnson, a Black woman, was the first to calculate a trajectory for NASA's missions to space and the moon, which students are also challenged to calculate. Students apply high-level thinking as they analyze, reason, explain, and justify their responses. These tasks honor the accomplishments and experiences of people from diverse backgrounds to fuel the belief that all people can do mathematics, regardless of culture, race, identity, or gender.



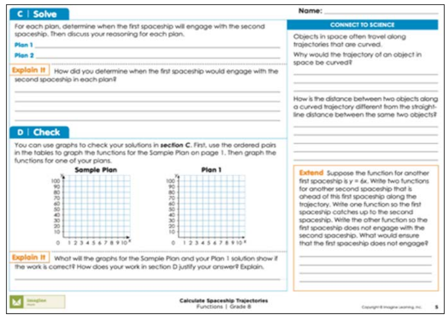


Figure 187. *Calculate Spaceship Trajectories*

IM 3+ recognizes no two students learn the same way; therefore, instruction is adapted so that all students can engage with grade-level content. **IM 3+** integrates **UDL** principles to provide learning opportunities that are within each student's zone of proximal development. To illustrate how **UDL** principles are integrated into a lesson, Figure 188 displays a problem in which students are asked to explore the concept of dividing by powers of 10.

- **Multiple Means of Representation**—Lessons presents information in multiple ways. These representations include visual text, English and Spanish audio, visual models, digital manipulatives, and an interactive glossary. For instance, if students are unclear what the vocabulary word "equally" means, they can access the interactive glossary for a definition, example, and visual representation (Figure 189). If students need to physically manipulate the problem, they can utilize **Imagine Math's** digital manipulatives (Figure 190).
- **Multiple Means of Action and Expression**—Lessons provide multiple opportunities for students to express and communicate their understanding. Students learn to **express** their mathematical thinking in a variety of ways, including drop-down menus (as shown in Figure 188), multiple-choice responses, drag-and-drop responses, virtual manipulatives, and the printable Journaling Pages (Figure 191).
- **Multiple Means of Engagement**—Lessons embed multiple strategies to **engage** students. Lessons provide problems that capture and maintain students' attention. Students can monitor their dashboard to keep track of their progress (Figure 192). Each student's customized learning pathway differentiates instruction to ensure learning is challenging but doable. Scaffolded support and informative feedback encourage positive mindsets and perseverance, while earning "Think! Points" increases motivation.

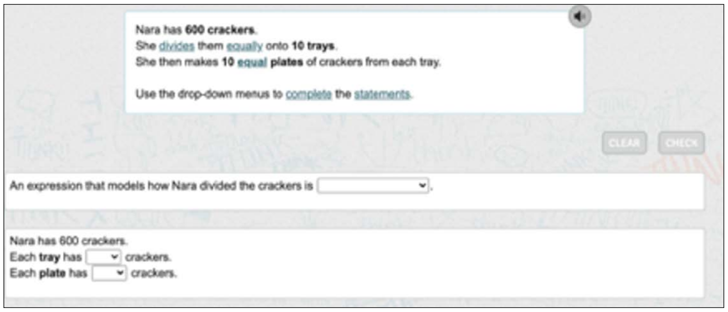


Figure 188. *G.5-Item 33686*

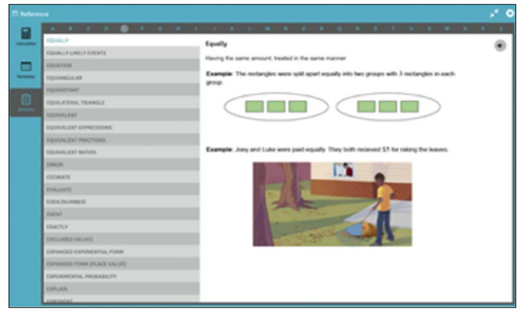


Figure 189. *Interactive glossary*

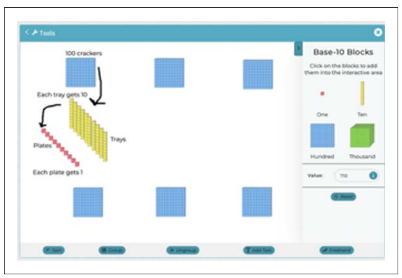


Figure 190. *Digital manipulatives*

Figure 191. *Journaling pages*

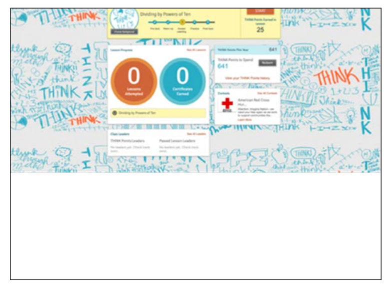


Figure 192. *Student dashboard*

IM 3+ makes learning **accessible** to all students by utilizing computer-based **scaffolding** to adapt instruction to students' individual learning needs. **IM 3+** believes all students are capable of success with grade-level content. Therefore, learning is scaffolded up appropriately, and never watered down, to maintain the rigor of each lesson. A unique array of adaptive scaffolds is integrated to carefully balance the level of challenge and support provided to students.

- **Language Support.** Students' home language is viewed as a valuable attribute. Throughout the lessons, students can listen to the lessons in English or Spanish and interact with teachers in either language. Students have access to academic and mathematics vocabulary words in English, Spanish, Tagalog, Haitian-Creole, Arabic, and Vietnamese. Providing support in students' home language helps reduce the cognitive load and allows them to focus on the key mathematical concept.
- **Multimedia Support.** Students have access to digital manipulatives (e.g., area models), references (e.g., formulas), video clips that provide informative feedback, multimedia response options (e.g., drag and drop), audio support (feedback provided verbally), and visual support (e.g., text highlighting).
- **Journaling Pages.** Journaling Pages are a printable offline resource that accompanies each lesson, designed to help students think through their problem-solving process. Students are encouraged to use these to record their thinking in their home language, while also making connections to content in English.
- **Immediate Feedback.** Immediate, informative feedback is provided during the Guided Learning

phase of each lesson. Feedback is designed to reinforce correct responses, address misconceptions or erroneous thinking, and provide scaffolded support. Figure 193 provides an example of the type of feedback that is provided to students. The use of text highlighting and labels helps students chunk the pertinent information and identify what information is still needed. This helps students determine the unknown quantity in the problem and how to use a variable to represent this quantity.

- **Math Helps and Live Teachers.** In the Guided Learning phase of each lesson, students can access two Math Help tabs, which provide guidance on how to solve the problem (Figure 194). If students need more intensive support, they can interact with a Live Teacher. Figure 195 displays a live conversation between a teacher and student. Notice the types of probing questions the teacher asked to gather insight on what the student understands (*"Explain to me the steps you have taken so far to begin to solve the problem."*) and the feedback that is provided (e.g., *"That's a great strategy! Thank you for your hard work."* and *"That's a great question. What do you think that means?"*).

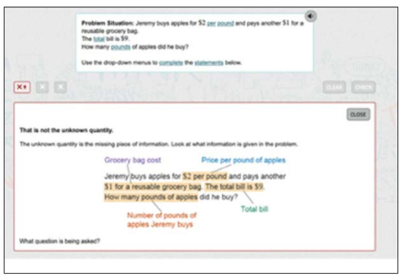


Figure 193. *G.4-Item 32999*

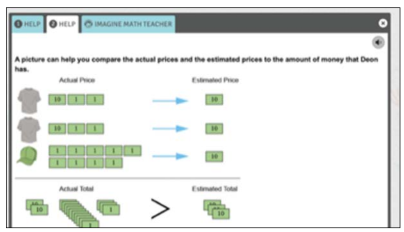


Figure 194. *G.3-Item 4596*

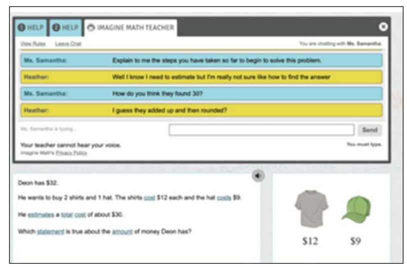


Figure 195. *Conversation with IM 3+ Live Teacher*

Principle 5: Utilize intrinsic and extrinsic motivational strategies to foster active engagement, collaboration, and perseverance.

WHAT THE RESEARCH SAYS:

Positive **self-efficacy** (Bandura, 2012) and motivation (Skaalvik et al., 2015) are strong predictors of mathematics achievement (Lewis et al., 2012; Parker et al., 2014). When students have positive beliefs about their capabilities (e.g., mathematics self-efficacy), they have greater confidence and are more likely to succeed in a task (Bandura, 2012). However, when students struggle in mathematics, their self-efficacy and motivation can diminish. This can decrease proficiency and interest in learning mathematics, while increasing anxiety and fear of failure (Gersten et al., 2009). Student **motivation** and **engagement** are thought to underpin overall success or failure in mathematics (Daly et al., 2019); therefore, it is important to consider how to effectively support students' self-efficacy and motivation. After all, their beliefs about their own competencies shape their habits of thinking, which go on to serve them throughout their life (Pajares, 2003).

Research has shown that the motivational attributes embedded in digital games are a powerful vehicle for transforming learning and engagement (Connolly et al., 2012). **Intrinsic motivation** refers to the act of doing something based on internal curiosity, interest, or inherent satisfaction (Filsecker & Hickey, 2014; Ryan & Deci, 2000). Research has found that students who are intrinsically motivated perform at higher levels (Lemos & Verissimo, 2014), are more inclined to persevere when faced with challenges (Huang, 2011), and develop a deeper understanding of content (Zainuddin et al., 2020). Students with intrinsic motivation also tend to set meaningful goals, monitor their progress toward achieving those goals (Bandura, 2012; Liao et al., 2019), and experience satisfaction mastering new mathematics concepts (Elliot & Harackiewicz, 1996). **Extrinsic motivation** reflects one's desire to engage in a behavior that is incentivized or produces an external reward (Moos & Marroquin, 2010). External motives can promote students' willingness to learn (Cameron, 2001; Theodotou, 2014) and verbal rewards can positively influence task completion (Marinak & Gambrell, 2008). Providing engaging content that promotes self-efficacy and motivation to learn not only influences students' achievement, but also fosters a love for learning.

RESEARCH-BASED RECOMMENDATIONS:

- Improve students' **self-efficacy** and **motivation** by capitalizing on student choice and interest, providing the appropriate level of task difficulty, incorporating frequent and focused feedback, modeling strategies like self-monitoring, encouraging effort and positive mindsets, and emphasizing mastery (Liao et al., 2019; Margolis & McCabe, 2006).
- Increase students' **intrinsic motivation** by incorporating real-world contexts that encourage curiosity and exploration through mastery-oriented quests and challenges (Alsawaier, 2018). This increases students' feelings of competence, autonomy, and relatedness through choice and self-directed learning (Deci & Ryan, 2008; Ng, 2018; Nicholson, 2015).
- Bolster students' **extrinsic motivation** by embedding external reward systems that provide continuous feedback (e.g., earned points) and opportunities to improve (Filsecker & Hickey, 2014).
- Optimize students' mathematics **motivation, engagement,** and **achievement** (Bai et al., 2020; Zainuddin et al., 2020) by integrating interactive gamified elements (e.g., points, badges, leader boards, trophies, customized avatars, and narratives).
    - Immediate feedback (verbal or in the form of points) can reinforce a growth mindset, or the idea that talents and abilities can be developed through effort, good teaching, and persistence (Mueller & Dweck, 1998; O'Rourke et al., 2014).
    - Customized avatars make learning fun, thereby improving intrinsic motivation, effort, and enjoyment of mathematics (Birk et al., 2016).
    - Digital games, badges, or points can boost extrinsic motivation by providing students with evidence of their capabilities and achievement (Gibson et al., 2015). They can also support internal motivation by encouraging students to monitor their goals, progress, and overall performance (Bai et al., 2020; Gnauk et al., 2012).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ improves students' **self-efficacy** by encouraging positive beliefs about their abilities in mathematics. Lessons help students develop confidence by providing immediate feedback that encourages effort. For instance, feedback praises effort like, *"Nice job. It can take a lot of work to find all of the right choices."* When students interact with a Live Teacher, they receive individualized instruction that reinforces mastery of grade-level content. When discussing student misconceptions, teachers use phrases like, *"I like your thinking," "Take some time to think about it and type your answer when you are ready!"* and *"See if you can use what we did to try this on your own. You can do this!"* This type of feedback attends to the learner's mathematical thinking, rather than the correctness of their response. It encourages them to take time to think through their answer, while promoting confidence in their abilities to do so independently.

IM 3+ comprises a unique **motivation** system based on a single idea—rewarding effort and accomplishment. This program encourages students to engage with challenging activities to promote perseverance and deep, meaningful learning. The program recognizes that students are motivated in different manners. To reach all learners, **IM 3+** promotes both **intrinsic** and **extrinsic motivation** by incorporating real-world situations that spark curiosity and interest, provide choice, foster feelings of relatedness, and build students' confidence. Each time a student engages in a mathematics activity, they can earn "Think! Points," which can be used to design an avatar, contribute to a classroom goal, or donate to a charity. In addition to earning "Think! Points," students acquire badges, which become competitive emblems that demonstrate students' practice, proficiency, and mathematical achievement.

IM 3+ encourages **intrinsic motivation** by allowing students to use earned "Think! Points" to customize their own avatar (Figure 196) and personalize their dashboard (Figure 197). Students make personalized design choices that allow them to own the process. Students choose from a variety of characters, accessories, and attributes. Many new avatar pieces were added to show more diversity and inclusion (Figure 198). **IM 3+ extrinsically motivates** students by embedding external reward systems using "Think! Points" and leaderboards (Figure 199) to drive positive learning behaviors.

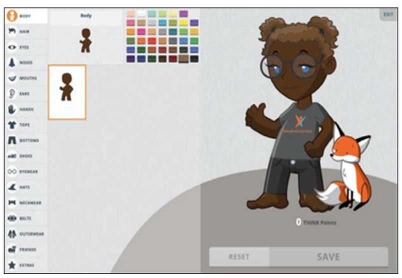


Figure 196. *Customizable avatar*

Figure 197. *Student dashboard*

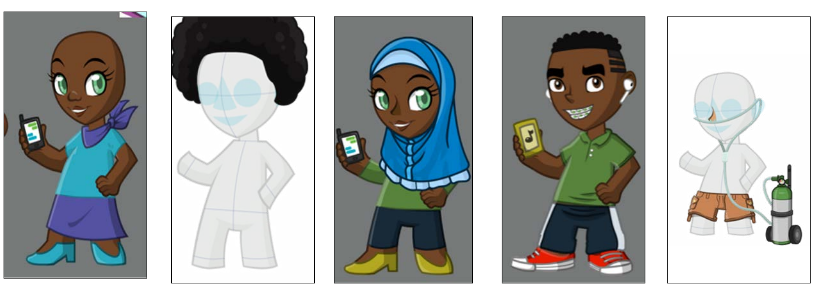


Figure 198. *Avatar characters, accessories, and attributes*

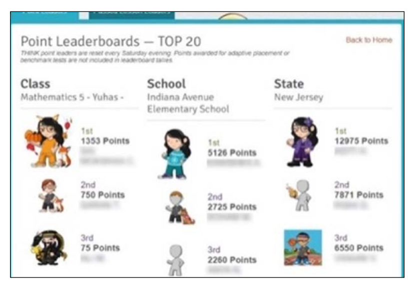


Figure 199. *Leaderboards*

To drive interest and **engagement**, students can earn points by participating in national and statewide contests that target a range of relevant and engaging themes, such as the "Ready, Set...Solve!" (Figure 200). This fun contest kicks off the 2021–2022 school year by motivating teachers to set up their classes and encouraging all students to engage in the mathematics lessons. Students who pass at least two math lessons at any point during the month will receive 5,000 bonus points to shop in the Avatar Store, donate to their Classroom Goal, or donate to the featured charity of the month.

Students can work independently or collaboratively as a team to earn points and donate them to special events or charities, such as Feeding America (Figure 201) or Boys and Girls Club of America (Figure 202).

Imagine Math contributes $1 for every 5,000 points donated. Engaging in activities that can make a positive impact on society promotes collaboration and community among all learners.

Figure 200. *Ready, Set, Solve!*

Figure 201. *Feeding America*

Figure 202. *Boys & Girls Club of America*

Principle 6: Differentiate instruction by offering informative feedback and adaptive assessments, while providing actionable data to inform mathematics teaching and improve student performance.

WHAT THE RESEARCH SAYS:

Experts in teaching and learning note that **differentiated instruction** occurs when the content of what a student is learning is adjusted in relation to their readiness to learn, interests, or ability profile (Hall et al., 2012; Moon, 2016; Subban, 2006; Tomlinson, 2014; Watts-Taffe et al., 2012). Research shows that students who receive adaptive instruction perform better on standardized mathematics assessments than their peers who receive nonadaptive methods of instruction (Aleven et al., 2017; Alshammari et al., 2016; Ma et al., 2014; VanLehn, 2011; Ysseldyke & Tardrew, 2007). Technology has become pivotal for delivering differentiated instruction that is understandable, engaging, relevant, and motivating (Walkington & Sherman, 2013). Technology provides each student with an adaptive learning pathway, which offers the support needed to master a concept or skill, such as scaffolding and immediate feedback (Belland et al., 2017; Gersten et al., 2009; Hattie & Timperley, 2007; Hudson et al., 2006; Smith et al., 2016).

The importance of ongoing assessment is widely acknowledged as a crucial part of instruction (Van Der Kleij et al., 2015). Formative and summative assessments are important measures for improving student learning. **Summative assessments** measure performance on an outcome measure, whereas **formative assessments** involve diagnosing students' learning needs and adjusting instruction to improve their performance (Schoenfeld, 2015). Formative assessments are considered essential for monitoring student progress, helping teachers make instructional decisions, and improving student achievement (Dalby & Swan, 2019; Faber et al., 2017; Hattie, 2009; NCTM, 2014b; Wang et al., 2019; Wiliam & Leahy, 2015; Xie et al., 2019). When teachers use data to identify students' strengths, areas of difficulty, interests, and aptitudes (Black & William, 1998; Kingston & Nash, 2011; Lai & Schildkamp, 2013; Wang et al., 2019; Xie et al., 2019), they can make informed decisions about their instruction in order to best support their students (Faber et al., 2017).

RESEARCH-BASED RECOMMENDATIONS:

- Optimize mathematics learning by providing adaptive, **differentiated instruction**. Instruction should provide appropriate lessons that fall within each student's zone of proximal development, consider their interests, and allow them to work at their own pace (Morgan, 2014; Tomlinson, 2014). Incorporate strategic scaffolds and immediate feedback to affirm correct responses, address misconceptions, and promote problem-solving strategies (Belland et al., 2017; Mitrovic et al., 2013; Van der Kleij et al., 2015).
- Integrate **summative** and **formative assessments** to continuously monitor student progress and growth, understand their thinking, and improve student achievement (Dalby & Swan, 2019; Schoenfeld, 2015).
- Use information gathered from assessments to make **data-driven decisions** about instruction and help students attain mastery of grade-level concepts (Faber et al., 2017).

HOW IMAGINE MATH 3+ INTEGRATES THESE RECOMMENDATIONS:

IM 3+ differentiates instruction by providing each student with a personalized learning pathway. Figure 203 illustrates the trajectory of an adapted pathway based on a student's results from Benchmark Test 1. A fifth-grade student is performing below grade level. The student is automatically assigned a fourth-grade precursor lesson designed to help them develop the necessary background knowledge before engaging in grade-level content. This student's pathway continues to adapt depending on their real-time performance. Relatedly, a teacher can adjust a student's pathways to a higher grade level if they are exhibiting advanced performance when tested on grade-level skills and concepts.

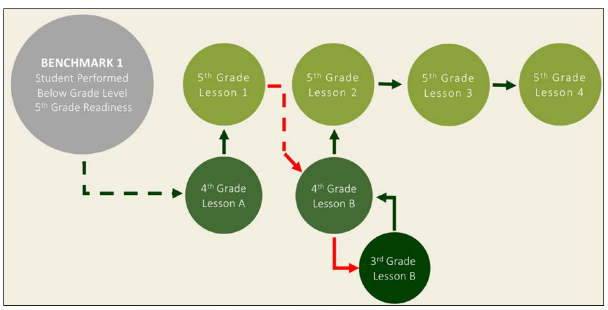


Figure 203. *Personalized learning pathway*

IM 3+ also differentiates instruction by providing scaffolded feedback after students respond to all non-assessment activities. When students answer a question correctly, they receive praise, motivation, points, and reinforcement of concepts with multiple representations. If students answer incorrectly, they receive verbal and visual indicators that the answer is incorrect. They are provided with explanations of the concepts, including visual diagrams and models to address misconceptions and prompt self-correction.

IM 3+ integrates **summative assessments** to assess student growth. The integrated Benchmark series (based on MetaMetrics' Quantile Framework for Mathematics) includes three 30-item adaptive tests designed to place students and measure student growth and progress. Figure 204 provides an overview of this series. The result of Benchmark Test 1 is a Quantile Measure, an **IM 3+** performance level (Figure 205), and an instructional grade level. A student's pathway is customized based on their performance on this Benchmark Test. Two additional Benchmark Tests are scheduled over the course of a school year and will adjust the content a student receives in their pathway as needed. These data provide information on what concepts the student has mastered, as well as any gaps the student needs to close to demonstrate proficiency.

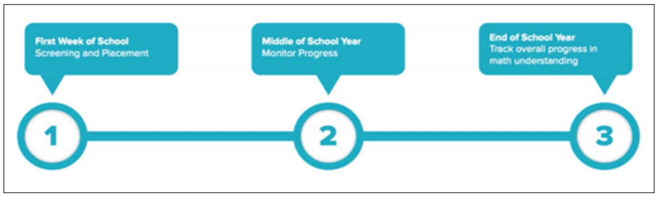
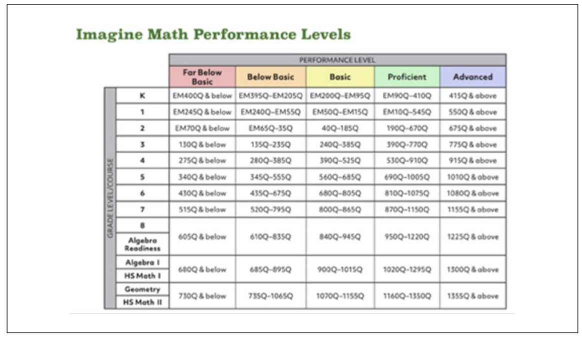


Figure 204. *Overview of Benchmark Test series*



Imagine Math Performance Levels

GRADE LEVEL/COURSE	PERFORMANCE LEVEL				
	Far Below Basic	Below Basic	Basic	Proficient	Advanced
K	EM400Q & below	EM395Q–EM205Q	EM200Q–EM95Q	EM90Q–410Q	415Q & above
1	EM245Q & below	EM240Q–EM55Q	EM50Q–EM15Q	EM10Q–545Q	550Q & above
2	EM70Q & below	EM65Q–35Q	40Q–185Q	190Q–670Q	675Q & above
3	130Q & below	135Q–235Q	240Q–385Q	390Q–770Q	775Q & above
4	275Q & below	280Q–385Q	390Q–525Q	530Q–910Q	915Q & above
5	340Q & below	345Q–555Q	560Q–685Q	690Q–1005Q	1010Q & above
6	430Q & below	435Q–675Q	680Q–805Q	810Q–1075Q	1080Q & above
7	515Q & below	520Q–795Q	800Q–865Q	870Q–1150Q	1155Q & above
8 / Algebra Readiness	605Q & below	610Q–835Q	840Q–945Q	950Q–1220Q	1225Q & above
Algebra I / HS Math I	680Q & below	685Q–895Q	900Q–1015Q	1020Q–1295Q	1300Q & above
Geometry / HS Math II	730Q & below	735Q–1065Q	1070Q–1155Q	1160Q–1350Q	1355Q & above

Figure 205. *Chart sourced Fall 2021*

IM 3+ also capitalizes on the use of **formative assessments** to continuously monitor student progress. In **IM 3+**, students take a quiz before and after each lesson to measure their understanding of that lesson's concept or skill. The program adapts instruction based on students' performance on these quizzes. Learners who earn at least 70% on the quiz will move on to the next lesson in their prescribed pathway. Students earning less than 70% will automatically be assigned additional lessons (when available), which are designed to help improve their understanding of this concept before reattempting this lesson at a later time. Teachers can also adjust a student's pathway to a higher grade level if they exhibit advanced performance when tested on grade-level skills and concepts.

IM 3+ recognizes that data should be actionable and a driving force for instruction. The Teacher Dashboard allows teachers to manage students, classes, and pathways, as well as view reports. The embedded reporting offers overviews of students' usage and classroom performance, helping educators identify performance patterns while tracking usage over the course of the school year. This includes:

- *Overview Report*—Monitor cumulative work per class, per student
- *Usage Report*—Information about students' use of the program, how individual students are progressing toward a goal, and their performance on lessons
- *Student Activity Report*—Information on how students are using the program and their behavior (e.g., clicking randomly, repeating the same lesson, or progressing as intended)
- *Student Progress Report*—Information on students' progress in their adaptive learning pathways
- *Mastery Report*—Information about the number of students failing, struggling, and passing individual standards and individual content strands. This information could be used to develop flexible groups.
- *Benchmark Growth Report*—Information on classroom averages or individual student performance levels. Allows teachers to track Quantile growth as benchmark assessments are administered throughout the year.
- *Leaderboard for Tracking Contests*—Information on the number of points each student has earned for the current week

Conclusion

Imagine Math recognizes the influential role mathematics plays in students' academic success and future career trajectories. While classrooms are becoming increasingly academically diverse, educators are doubling down on their efforts to implement instruction that meets students' unique learning needs. With advances in technology, **Imagine Math** capitalizes on the use of a digital learning environment to differentiate instruction, promote equity, and empower educators to make data-based instructional decisions.

Imagine Math, a powerful supplemental program, is grounded in research and guided by a well-specified theory of action. This theory of action explains how **Imagine Math** can improve students' mathematics achievement. By intentionally translating the most robust mathematics research into practice, **Imagine Math** helps students develop the foundational skills needed to succeed. The program's innovative design of developmentally appropriate learning environments, adaptive learning pathways, and unique motivational elements not only accelerate learning, but also help students develop into curious, confident, and competent mathematicians. Rigorous and relevant mathematics lessons ensure all students have access to opportunities to develop a deep, conceptual understanding of grade-level content. With more than 3.4 million students enrolled in Fall 2021, **Imagine Math** has become a transformational tool for building teachers' capacity to drive breakthroughs along every student's unique learning journey.

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